A Too$\lambda\kappa$it for Ri$\nu\gamma$-$\Lambda\Omega$E $\kappa\rho$y$\pi\tau o\gamma\rho\alpha\phi$

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[1]INRIA & ENS Paris

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Eurocrypt 2013
27 May

A Toolkit for Ring-LWE Cryptography

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  - ⋆ Signatures [LM'08,GPV'08,L'09,CHKP'10,B'10,GKV'10,BF'11ab,L'12,...]
  - ⋆ (H)IBE & FE [GPV'08,CHKP'10,ABB'10,AFV'11,...]
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- Most modern schemes are based on the SIS/LWE problems [A'96,R'05] and/or their ring variants [M'02,PR'06,LM'06,LPR'10].
  - ✗ SIS/LWE aren't quite practical: $\Omega(n^2)$ key sizes and runtimes
  - ✔ Ring-based primitives are! $\tilde{O}(n)$ complexity

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Ring $\boxed{R := \mathbb{Z}[X]/(1 + X^n)}$ for some $n = 2^k$, $\quad R_q := R/qR$.

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("Expansion factor" $\sqrt{n}$ is worst-case, often quite loose.)

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✔ Ring-LWE (appropriately defined) is hard in any cyclotomic [LPR'10]

…assuming problems on ideal lattices are quantum-hard in the worst case.

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Yuck!!!

✗ Irregular $\Phi_m(X)$ induces cumbersome, slower operations modulo $\Phi_m(X)$

✗ Large expansion factors — up to super-polynomial $n^{\omega(1)}$ [Erdős'46]

✗ Provable & concrete security also degrade with expansion factor: pay twice!

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2. In analysis, use canonical embedding to define geometry.
3. Use decoding basis of dual ideal $R^\vee$ for noise generation & decoding.
   - ✔ Corresponds to the "true" definition of ring-LWE.

Tensorial Decomposition and the “Powerful” Basis

- Recall: $\Phi_p(X) = 1 + X + \cdots + X^{p-1}$ and $\Phi_{p^e}(X) = \Phi_p(X^{p^{e-1}})$.

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Ancient Theorem [Kummer, 1840s]

- Let $m = \prod_\ell m_\ell$ be the prime-power factorization of m.

  Then the mth cyclotomic ring $R = \mathbb{Z}[X]/\Phi_m(X)$ is isomorphic to the tensor product of all the m_ℓth cyclotomic rings:

  $$R \cong \mathbb{Z}[X_1, X_2, \ldots]/(\Phi_{m_1}(X_1), \Phi_{m_2}(X_2), \ldots).$$

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- It is not the "power" basis $\{1, X, X^2, \ldots, X^{\varphi(m)-1}\}$ of $\mathbb{Z}[X]/\Phi_m(X)$.

  E.g., for $m = 15$ it's $\{X^j\}$ for $j \in \{0, 3, 5, 6, 8, 9, 11, 14\}$.

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Geometry: Norms, singular values, Gram-Schmidt orthogonalization, dual basis, etc. all behave well under tensoring.

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Algebra: Exposes fine-grained structure of the ring and its relationships with other cyclotomic rings.

E.g.: has applications in "ring-switching" [GHPS'12] and new bootstrapping [AP'13] algorithms for FHE.

Algorithms: Efficiently reduces all operations to the prime-power case, by dealing with each X_ℓ independently.

E.g.: simple, fast conversions to/from "evaluation (CRT) representation," via sequence of prime-power FFTs.

Geometry: Norms, singular values, Gram-Schmidt orthogonalization, dual basis, etc. all behave well under tensoring.

E.g.: powerful basis is better-conditioned than power basis.

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- ✔ Ring-LWE is provably hard with (spherical) Gaussian noise under σ.

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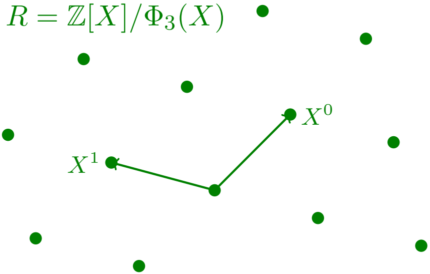
- $R = \mathbb{Z}[X]/\Phi_p(X)$ under embedding σ is a lattice in $\mathbb{C}^{p-1}$.



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- Its dual $R^\vee$ has $\mathbb{Z}$-basis $\{d_j\}$, given by $\langle \sigma(d_j)\,,\, \sigma(X^{j'}) \rangle = \delta_{j,j'}$.
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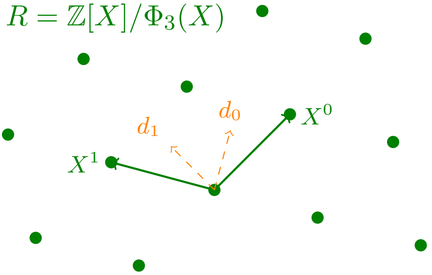

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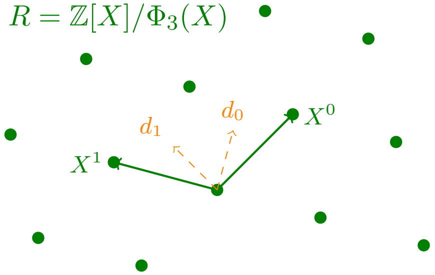
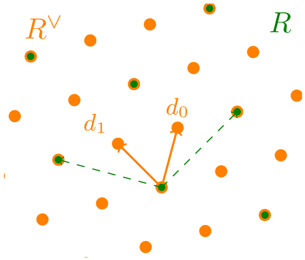
- $R^\vee$ is a (fractional) ideal, and $pR^\vee \subseteq R \subseteq R^\vee$, with $pR^\vee \approx R$.





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- By contrast, such optimal decoding is not possible for R/qR, because $R^\vee$ lacks optimally short elements for its density.
- **Bottom line:** using $R^\vee$ is actually beneficial in applications!

  (And "advanced" applications benefit even more from its algebraic properties.)

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Thanks!

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