Lattices that Admit Logarithmic Worst-Case to Average-Case Connection Factors

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Worst-case versus average-case complexity

Lattices are an intriguing case study:

- Believed hard in the worst case
- Worst-case / average-case reductions

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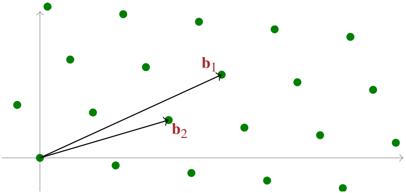
This Talk...

- Not (exactly) about crypto
- Special, natural class of algebraic lattices
- Very tight worst-case/average-case reductions
  - Much tighter than known for general lattices
- Distinctions between decision and search
- Many open problems

Lattices

Let $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset \mathbb{R}^n$ be linearly independent.
The n-dim lattice $\mathcal{L}$ having basis $\mathbf{B}$ is:

$$\mathcal{L} \quad = \quad \sum_{i=1}^{n} (\mathbb{Z} \cdot \mathbf{b}_i)$$



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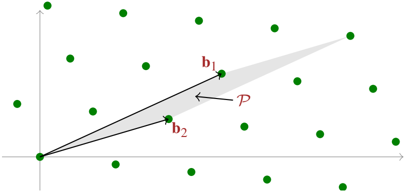

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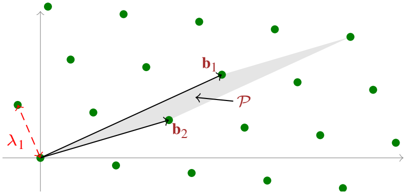

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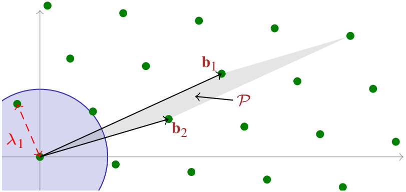

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Minkowski's Theorem

$$\lambda_1 \quad \leq \quad \sqrt{n} \cdot \mathrm{vol}(\mathcal{P})^{1/n}$$

(Non-constructive, non-algorithmic proof. . .)

Shortest Vector Problem (SVP)

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Algorithms for SVP$_\gamma$

- $\gamma(n) \sim 2^n$ approximation in poly-time [LLL]
- Can trade-off running time/approximation [Sch,AKS]

Worst-Case/Average-Case Connections [Ajtai,...]

For some $\gamma(n) = \mathrm{poly}(n)$ ("connection factor"):

SVP$_\gamma$ hard in the worst case

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Cryptographic Applications

- One-way & collision-resistant functions [Ajtai,GGH,...]
- Public-key encryption [AjtaiDwork,Regev]

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Optimizing the Connection Factor γ

- Interesting to characterize complexity
- Important for crypto due to time/accuracy tradeoff
- Current best $\gamma(n) \sim n$ [MicciancioRegev]

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- Decision-SVP is *easy* to approximate: $\lambda_1 \approx$ Minkowski bound.
  Not NP-hard!
- Search-SVP appears hard, despite structure.
  Best known algorithms [LLL,Sch,AKS].

Our Results

Complexity of Ideal Lattices

1. Connection factors as low as $\gamma = \sqrt{\log n}$.
   - Based on search-SVP. (Decision is *easy*.)
   - For SVP in any ℓ_p norm. (Stay for CCC.)

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Subtleties

No *efficient* constructions of best number fields (yet).

⇒ Non-uniformity (preprocessing) in reductions.

⇒ Crypto is tricky.

⇒ Many interesting open problems!

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Structure used for functionality & efficiency.

Connection factors $\gamma \sim n$ or more.

Worst-to-Average Reduction [Ajtai,...]

Average-Case Problem

For uniform $\mathbf{a}_1, \ldots, \mathbf{a}_m \leftarrow \mathbb{Z}^n \bmod q$, find short nonzero $\mathbf{z} \in \mathbb{Z}^m$:

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Connection Factor

- Size of solution $\mathbf{z} \in \mathbb{Z}^m$
- Lengths of offset vectors $\nearrow_i$

Our Approach

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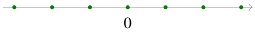

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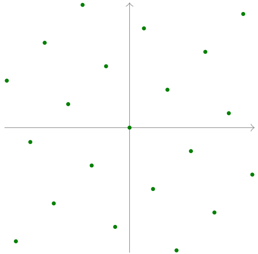

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Avg-case problem	for $\mathbf{a}_i \leftarrow \mathbb{Z}^n \bmod q$ find small $z_i \in \mathbb{Z}$: $\sum z_i \mathbf{a}_i = 0 \bmod q$	for $a_i \leftarrow \mathcal{O}_K \bmod q$ find "small" $z_i \in \mathcal{O}_K$: $\sum z_i a_i = 0 \bmod q$

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   - Ideal lattice primal & dual have (optimally) large λ_1.

Crash Course in Algebraic Number Theory

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Pretty Pictures: Ideal Lattices

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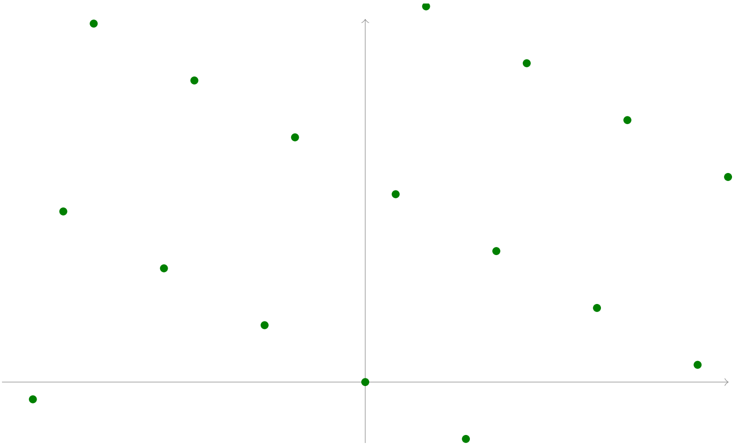

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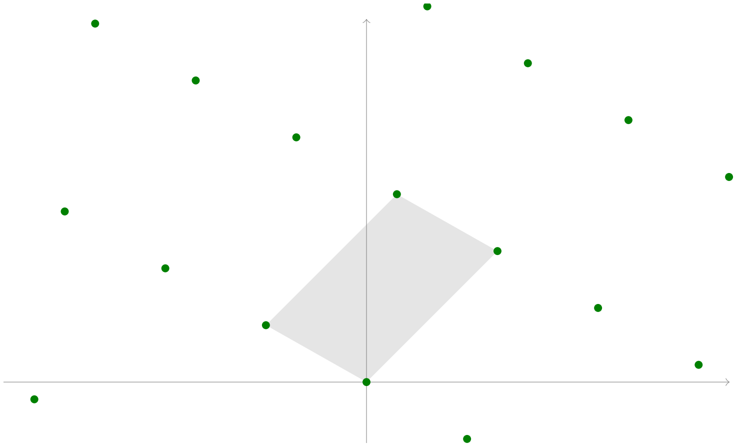
- Root discriminant $\mathcal{D}_K$ = (fundamental volume)$^{2/n}$



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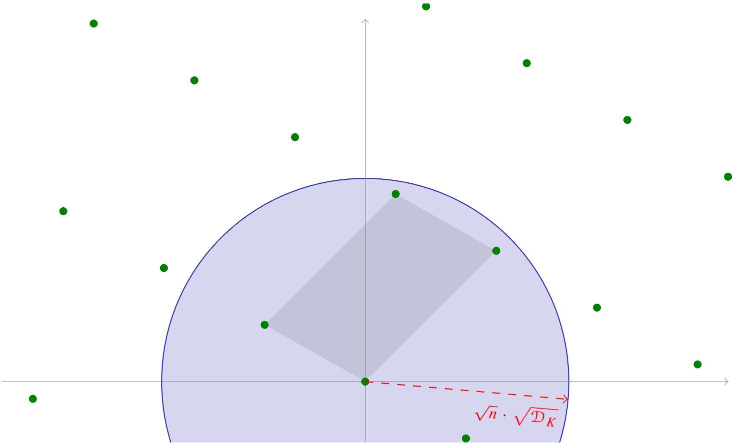

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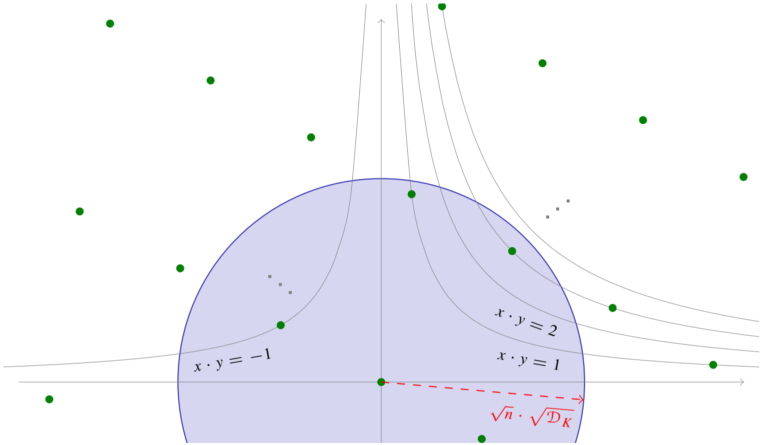

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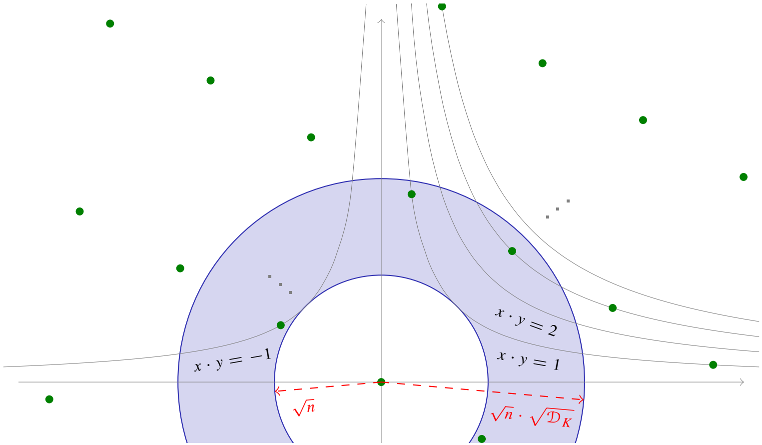

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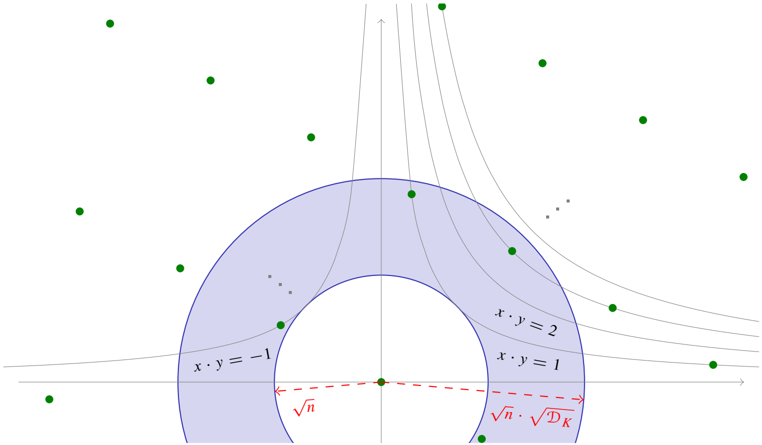
- Same for dual lattice $\Rightarrow$ short offsets $\nearrow$



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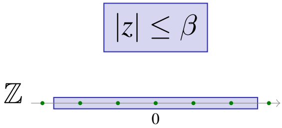

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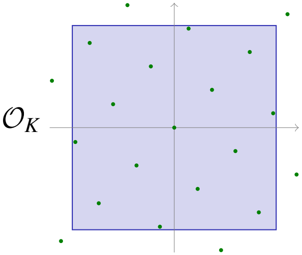

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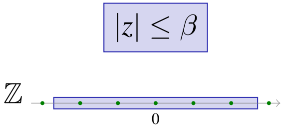

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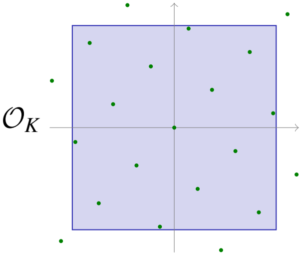

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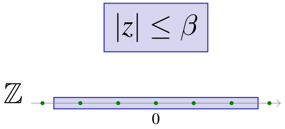

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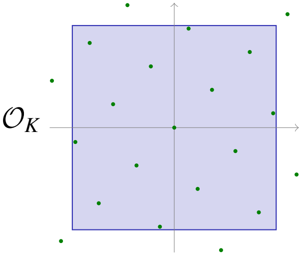

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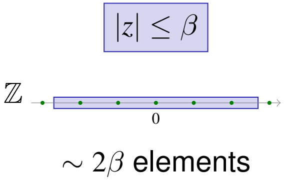

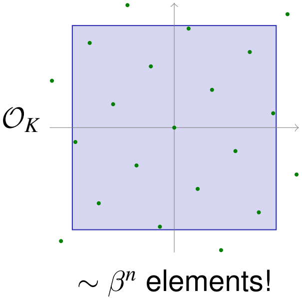

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- Denser $\mathcal{O}_K \Rightarrow$ denser, shorter solutions.

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3. Crypto is tricky: must map $\{0,1\}^*$ to short elts of $\mathcal{O}_K$.

   Q3: Can this be done efficiently?