Peculiar Properties of Lattice-Based Encryption

Chris Peikert
Georgia Institute of Technology

Public Key Cryptography
and the Geometry of Numbers

7 May 2010

Talk Agenda

Encryption schemes with special features:

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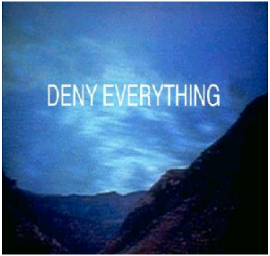
1. "(Bi-)Deniability"



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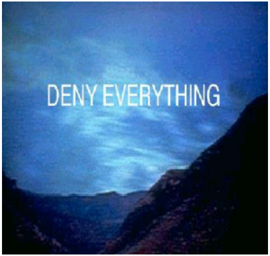
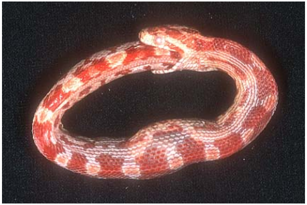


2. "Circular" Security



Part 1:

Deniable Encryption

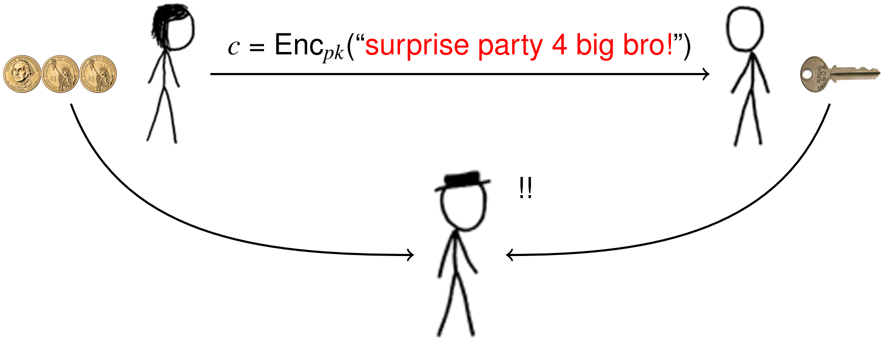
- A. O'Neill, C. Peikert (2010)
  "Bideniable Public-Key Encryption"

Deniable Encryption


c = Enc$_{pk}$("surprise party 4 big bro!")


(Images courtesy xkcd.org)

Deniable Encryption



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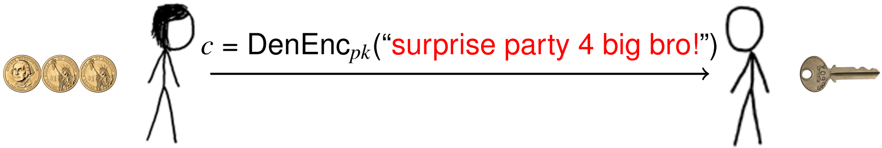

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What We Want

1. Bob gets Alice’s intended message, but …

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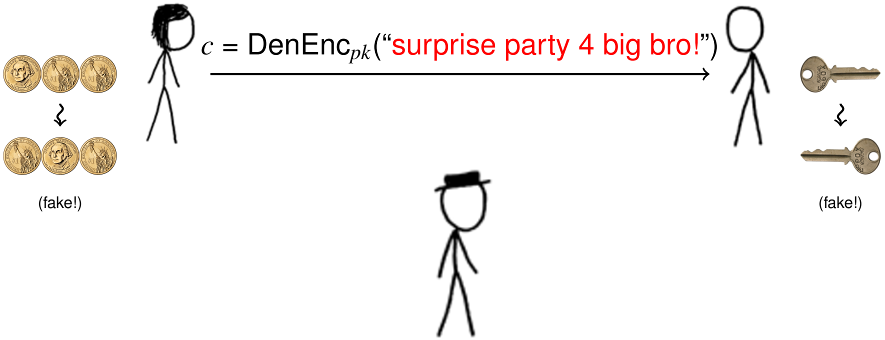

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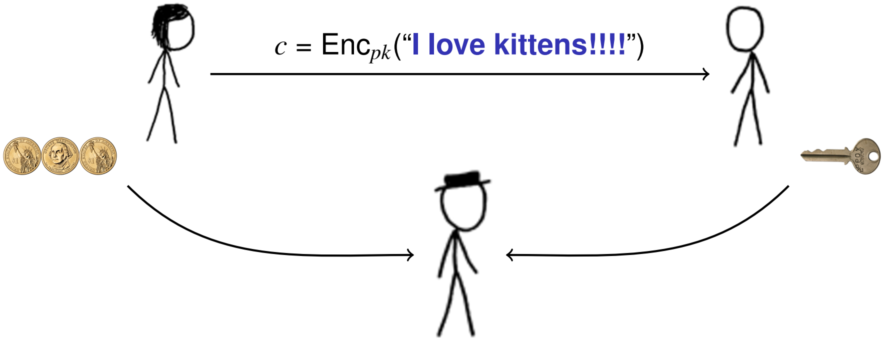

What We Want

1. Bob gets Alice's intended message, but . . .
2. Fake coins & keys 'look as if' another message was encrypted!

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Applications of Deniability

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1. Anti-coercion: ‘off the record’ communication (journalists, lawyers, whistle-blowers), 1984
2. Voting: can reveal *any* candidate, so can’t ‘sell’ vote (?)
3. Secure protocols tolerating *adaptive* break-ins [CFGN’96]

State of the Art

Theory [CanettiDworkNaorOstrovsky'97]

- Sender-deniable encryption scheme
- Receiver-deniability by adding interaction & switching roles
- Bi-deniability by interaction w/ 3rd parties (one must remain uncoerced)

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Practice: TrueCrypt, Rubberhose, ...

- Limited deniability: "*move along, no message here...*"

  Plausible for *storage*, but not so much for *communication*.

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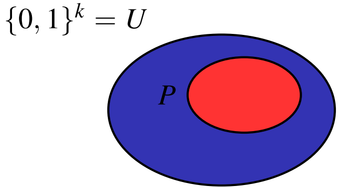
2. "Plan-ahead" bi-deniability with *short* keys
   - Bounded number of alternative messages, decided in advance

A Core Tool: Translucent Sets [CDNO'97]



Public description pk with secret 'trapdoor' sk.

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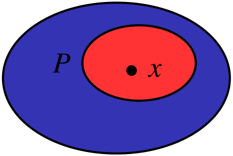
$\{0,1\}^k = U$



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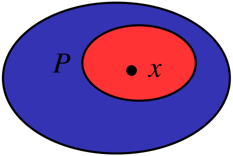

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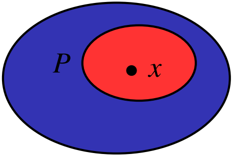

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- Many instantiations: trapdoor perms (RSA), DDH, lattices, ...

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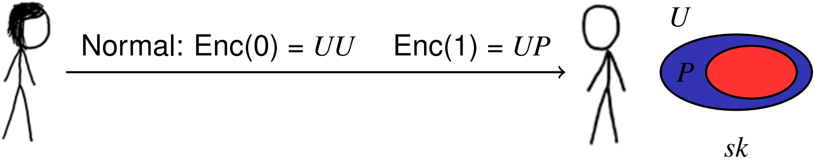

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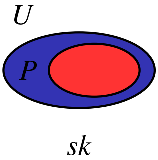
Normal: Enc(0) = UU Enc(1) = UP

Deniable: Enc(0) = PP Enc(1) = UP



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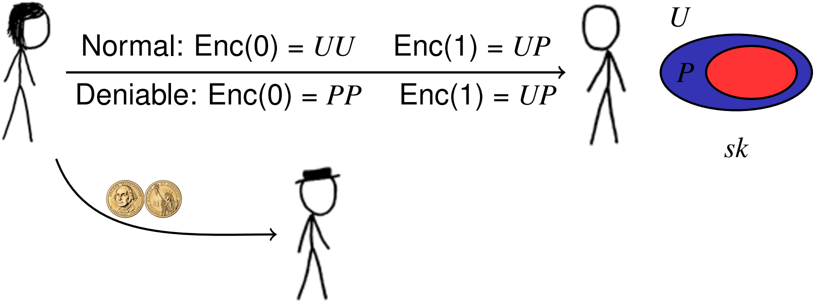

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✔ Alice can fake: $PP \to UP \to UU$

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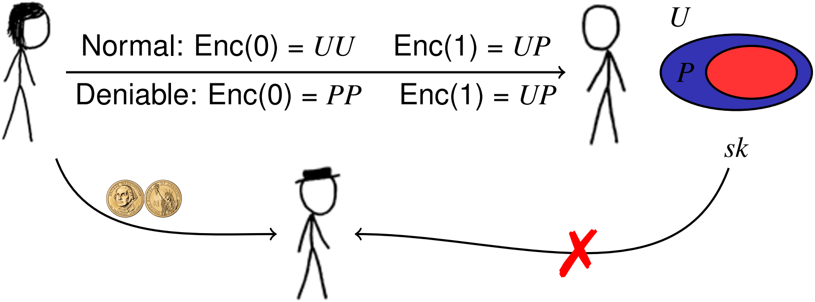
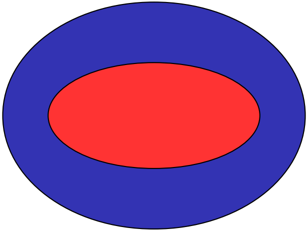


Deniability

- ✔ Alice can fake: $PP \to UP \to UU$
- ✗ What about Bob?? His sk reveals the true nature of the samples!

Our Contribution: Bi-Translucent Sets



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1. Each pk has many sk, each inducing a *slightly different* P-test.

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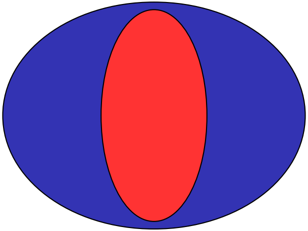

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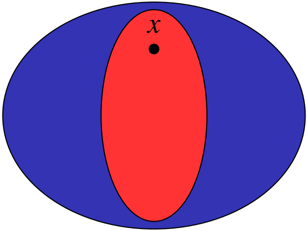

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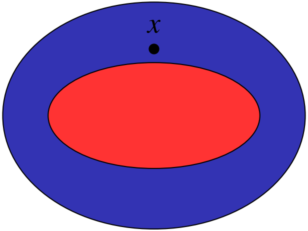

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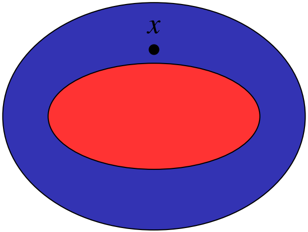

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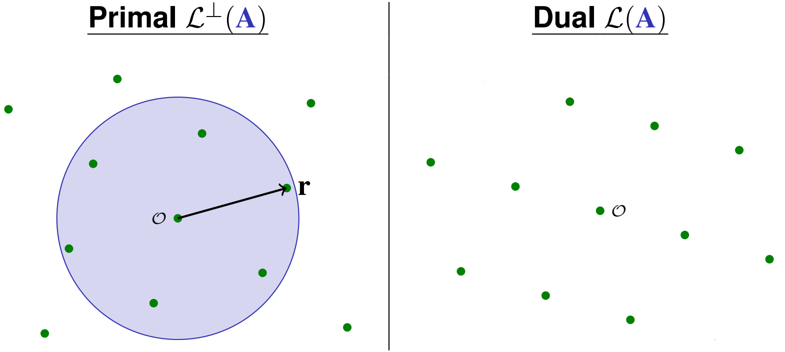
$\Rightarrow$ Bob can also fake $P \to U$!

Lattice-Based Bi-Translucent Set



Basic Translucency

- pk = parity check $\mathbf{A}$ of lattice $\mathcal{L}^{\perp}(\mathbf{A})$.
- sk = Gaussian (short) vector $\mathbf{r} \in \mathcal{L}^{\perp}$. (I.e., $\mathbf{A}\mathbf{r} = \mathbf{0} \in \mathbb{Z}_q^n$.)

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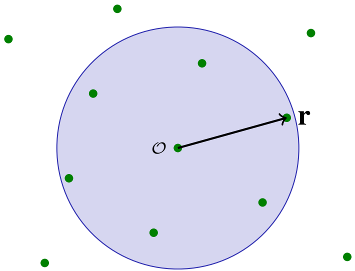

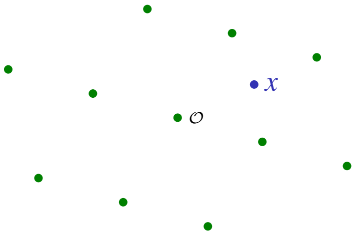

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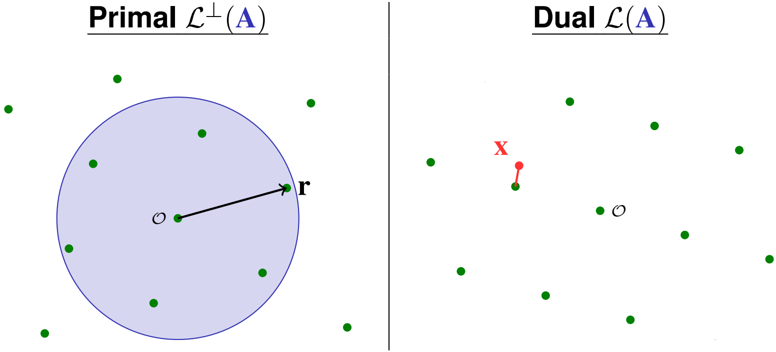

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- P-sample = $\mathbf{x} = \mathbf{A}^t\mathbf{s} + \mathbf{e}$ (LWE). Then $\langle \mathbf{r}, \mathbf{x} \rangle \approx 0 \bmod q$.

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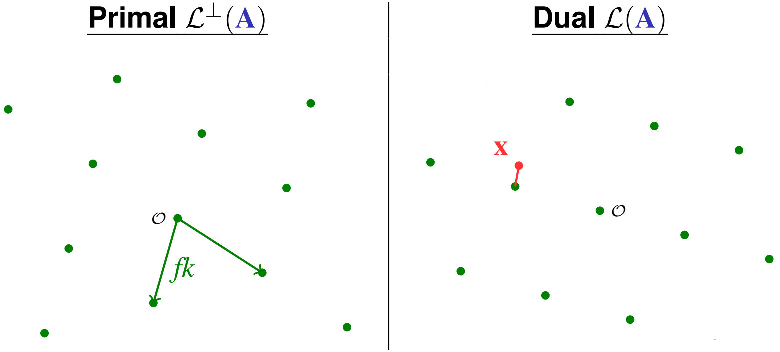

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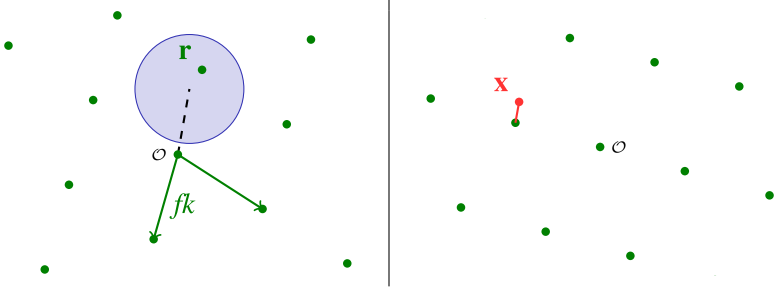

Receiver Faking

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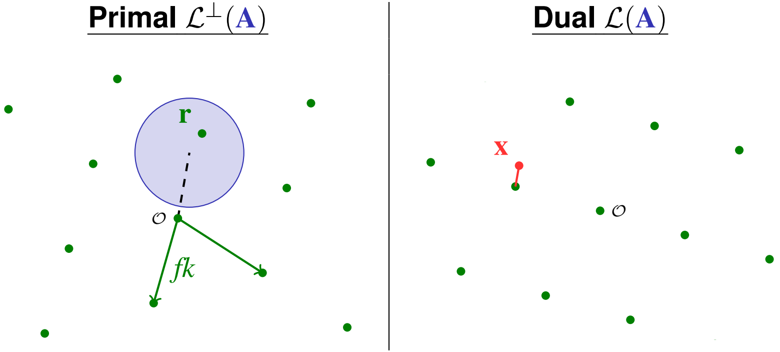
- Given P-sample $\mathbf{x}$, choose fake $\mathbf{r} \in \mathcal{L}^{\perp}$ *correlated* with $\mathbf{x}$'s error.

  Then $\langle \mathbf{r}, \mathbf{x} \rangle$ is uniform mod $q \Rightarrow \mathbf{x}$ is classified as a U-sample.

Lattice-Based Bi-Translucent Set



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- Fake $\mathbf{r}$ depends heavily on $\mathbf{x}$. Why would it 'look like' a 'normal' $\mathbf{r}$?

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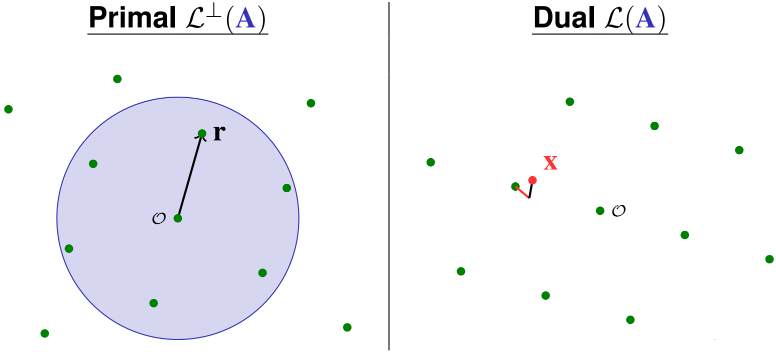

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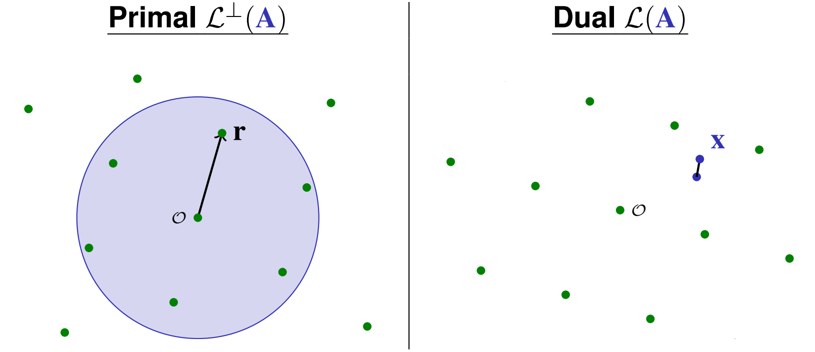

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- Finally, replace LWE with uniform $\Rightarrow$ normal $\mathbf{r}$ and U-sample $\mathbf{x}$.

Closing Thoughts on Deniability

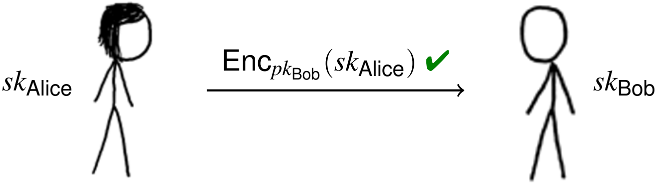
- Faking sk requires ‘oblivious’ misclassification (of P as U)
- Bi-deniability from other cryptographic assumptions?
- Full deniability, without alternative algorithms?

Part 2:

Circular-Secure Encryption

- B. Applebaum, D. Cash, C. Peikert, A. Sahai (CRYPTO 2009)
  "Fast Cryptographic Primitives and Circular-Secure Encryption Based on Hard Learning Problems"

Circular / “Clique” / Key-Dependent Security



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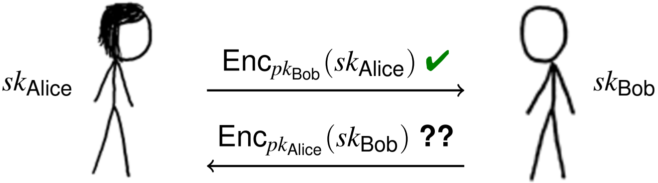

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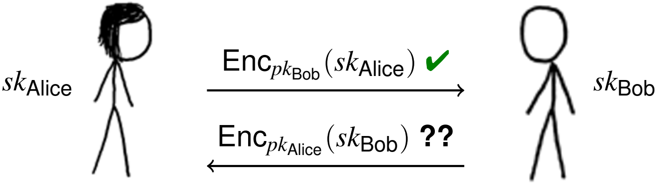

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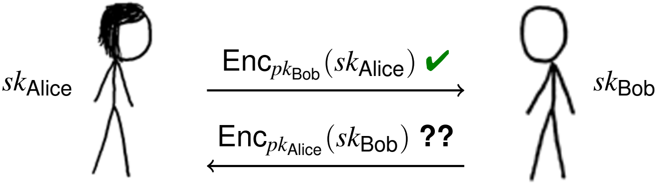

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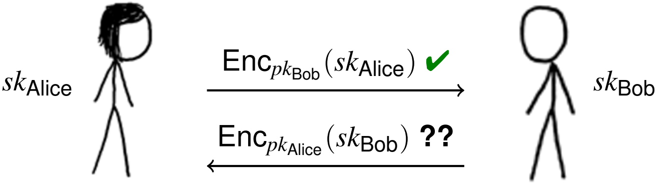

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- Some (semantically secure) schemes are actually circular-*insecure* [ABBC’10,GH’10]

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Public key	Enc Time	Ciphertext
k^2 group elts $\Downarrow$ k^3 bits	k expon $\Downarrow$ k^4 bit ops	$\geq k$ group elts $\Downarrow$ $\geq k^2$ bits

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- Security: same. Follows general [BHHO'08] approach.
- Efficiency: comes 'for free*' with existing schemes! [R'05,PVW'08]

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Regev's Cryptosystem

- Decision LWE problem: distinguish samples

$$(\mathbf{a}_i\ ,\ b_i = \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i) \in \mathbb{Z}_q^n \times \mathbb{Z}_q \quad \text{from} \quad \text{uniform } (\mathbf{a}_i\ ,\ b_i)$$

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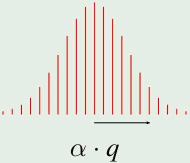
The Scheme

- Keys: $sk = \mathbf{s} \leftarrow \mathbb{Z}_q^n$,

$$pk = \left(\begin{matrix} \vdots \\ \mathbf{A}^t \\ \vdots \end{matrix} \right) \ , \ \left(\begin{matrix} \vdots \\ \mathbf{b} \\ \vdots \end{matrix} \right) = \mathbf{A}^t \mathbf{s} + \mathbf{e}$$



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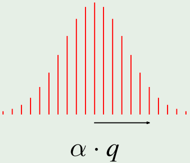
$$(\mathbf{a}_i\ ,\ b_i = \langle \mathbf{a}_i, \mathbf{s} \rangle + e_i) \in \mathbb{Z}_q^n \times \mathbb{Z}_q \quad \text{from} \quad \text{uniform } (\mathbf{a}_i\ ,\ b_i)$$

The Scheme

- Keys: $sk = \mathbf{s} \leftarrow \mathbb{Z}_q^n$,

$$pk = \begin{pmatrix} \vdots \\ \mathbf{A}^t \\ \vdots \end{pmatrix}\ ,\ \begin{pmatrix} \vdots \\ \mathbf{b} \\ \vdots \end{pmatrix} = \mathbf{A}^t\mathbf{s} + \mathbf{e}$$



- Encrypt: Let $(\mathbf{u} = \mathbf{A}\mathbf{r}\ ,\ v = \langle \mathbf{b}, \mathbf{r} \rangle)$ for $\mathbf{r} \leftarrow \{0,1\}^m$.
  For message $\mu \in \mathbb{Z}_p$ (where $p \ll q$), ciphertext = $(\mathbf{u}\ ,\ v + \mu \cdot \lfloor \frac{q}{p} \rceil)$.

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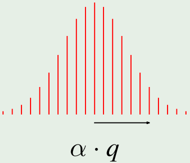

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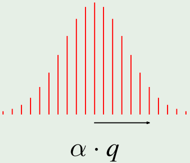

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- Security proof: uniform $pk = (\mathbf{A}, \mathbf{b}) \implies$ uniform ciphertext $(\mathbf{u}, v)$.

Self-Reference ?

An Observation

- With $(\mathbf{u} = \mathbf{A}\mathbf{r}\ ,\ v = \langle \mathbf{b}, \mathbf{r} \rangle)$, the ciphertext $(\mathbf{u}' = \mathbf{u} - \lfloor \frac{q}{p} \rfloor \cdot \mathbf{e}_1\ ,\ v)$ decrypts as $v - \langle \mathbf{u}', \mathbf{s} \rangle \approx (s_1 \bmod p) \cdot \lfloor \frac{q}{p} \rfloor$. (Or any affine fct of $\mathbf{s}$.)

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  And does $s_1 \in \mathbb{Z}_q$ 'fit' into the message space $\mathbb{Z}_p$? Also no!

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Modifying the Scheme

1. Use $q = p^2$ for divisibility. (Need new search/decision reduction for LWE.)

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Modifying the Scheme

1. Use $q = p^2$ for divisibility.
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   Then $(\mathbf{u}, v)$ is *itself* an $\mathsf{LWE}_{\mathbf{s}}$ sample*. [R'05,GPV'08]

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   ?? But is it secure to use such an $\mathbf{s}$??

LWE with Gaussian Secret

- Transform $\text{LWE}_{\mathbf{s}}$ (for arbitrary $\mathbf{s}$) into $\text{LWE}_{\mathbf{e}}$ for Gaussian secret $\mathbf{e}$:

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(\mathbf{a}, b) \quad &\mapsto \quad (\mathbf{a}' = -\mathbf{A}^{-1}\mathbf{a} \ , \ b + \langle \mathbf{a}', \mathbf{b} \rangle) \\
&= \quad (\mathbf{a}' \ , \ \langle \mathbf{a}, \mathbf{s} \rangle + e - \langle \mathbf{A}^{-1}\mathbf{a}, \mathbf{A}^t \mathbf{s} \rangle + \langle \mathbf{a}', \mathbf{e} \rangle) \\
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  (Also maps uniform samples $(\mathbf{a}, b)$ to uniform $(\mathbf{a}', b')$).

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Clique & Affine Security (Again, For Free)

- Repeating transform produces ind. sources $\text{LWE}_{\mathbf{e}_1}$, $\text{LWE}_{\mathbf{e}_2}$, ...
- Side effect: a *known affine relation* between *unknowns* $\mathbf{s}$ and $\mathbf{e}_i$.

  This lets us create $\text{Enc}_{pk_i}(\text{affine}(\mathbf{e}_j))$ for any i, j.

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Thanks!