Lattice-Based Cryptography: Mathematical and Computational Background

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crypt@b-it 2013

Lattice-Based Cryptography

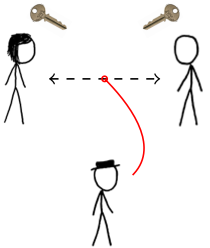
$y = g^x \bmod p$

$N = p \cdot q$

$m^e \bmod N$

$e(g^a, g^b)$

$\Longrightarrow$



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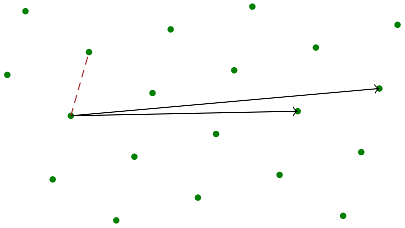

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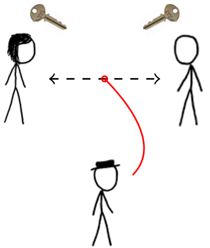

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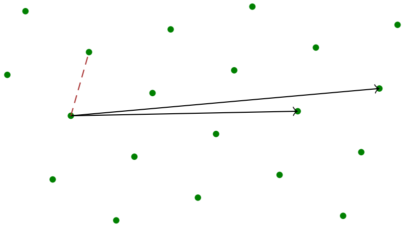

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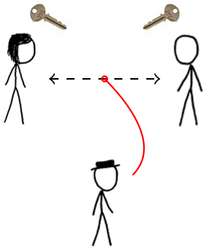

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- Simple description and implementation

(Images courtesy xkcd.org)

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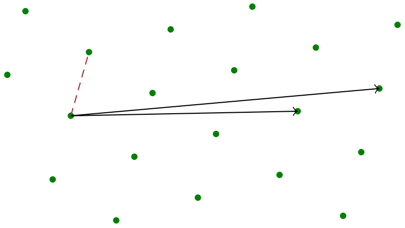

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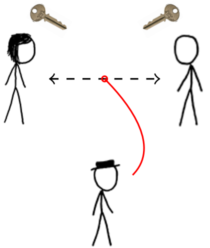

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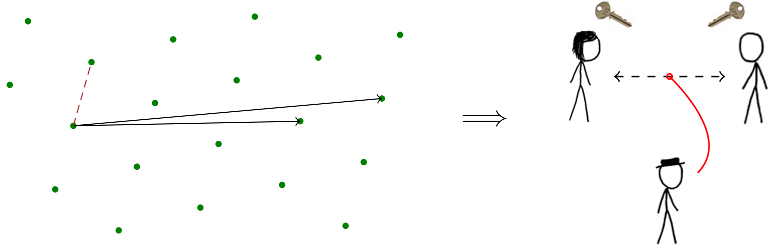

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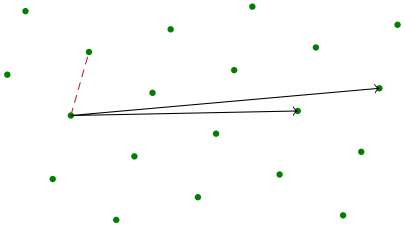

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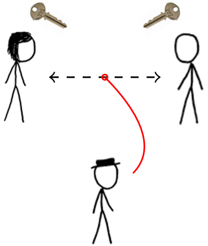

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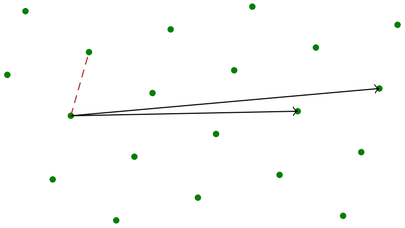

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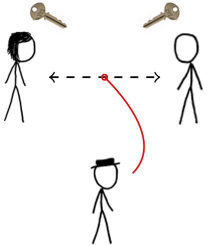

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- Security from worst-case assumptions [Ajtai96,...]
- Solutions to "holy grail" crypto problems [Gentry09,...]

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Part 1:

Mathematical Background

Coming up:

1. Definitions: lattice, basis, determinant, cosets, successive minima, ...
2. Two simple bounds on the minimum distance.

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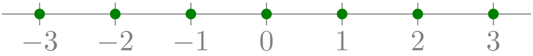

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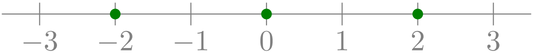

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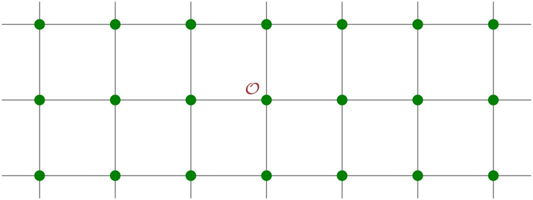

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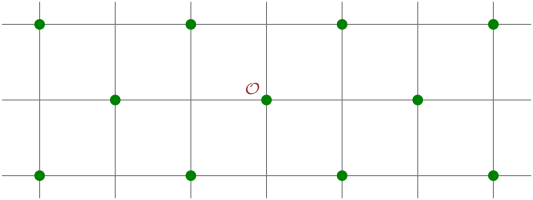

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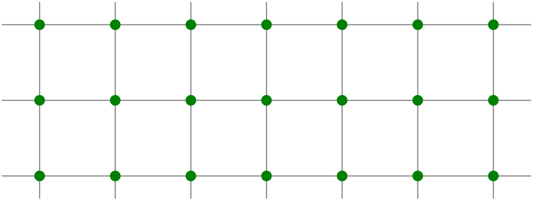

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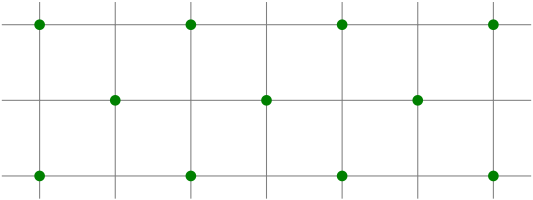

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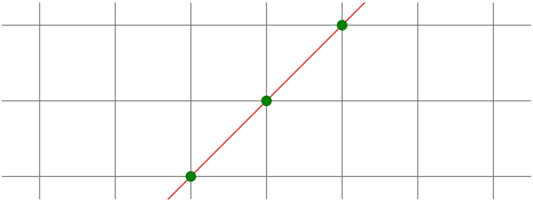

Representing Lattices: Bases

- Basis of $\mathcal{L}$: ordered set (i.e., matrix) $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ s.t.

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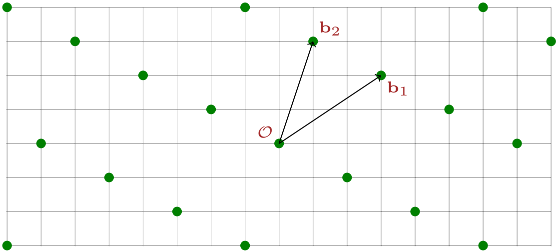

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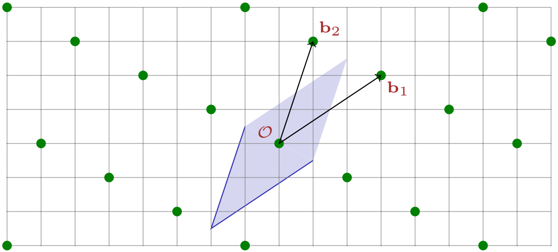

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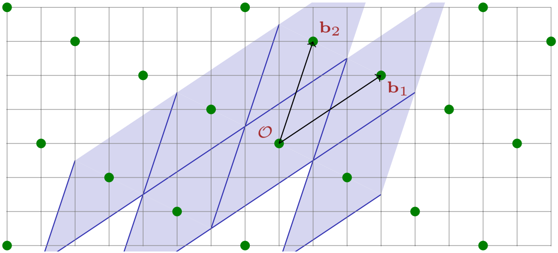

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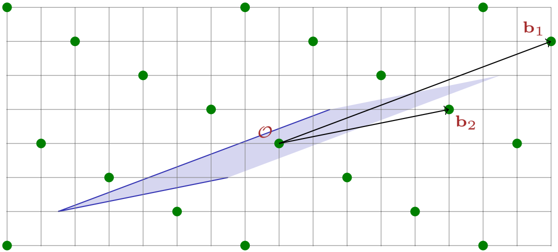
- A basis is not unique: $\mathbf{BU}$ is also a basis iff $\mathbf{U} \in \mathbb{Z}^{n \times n}, \det(\mathbf{U}) = \pm 1$.



Cosets and Determinant

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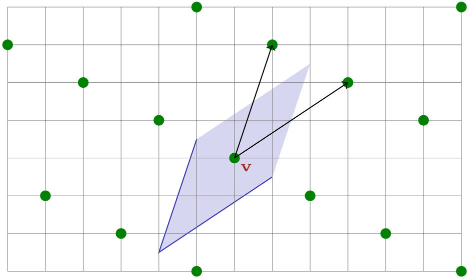

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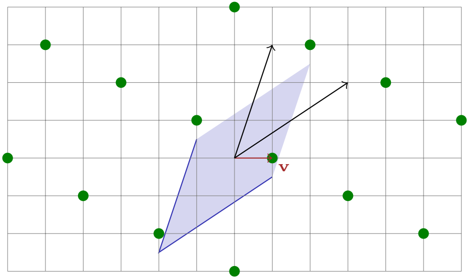

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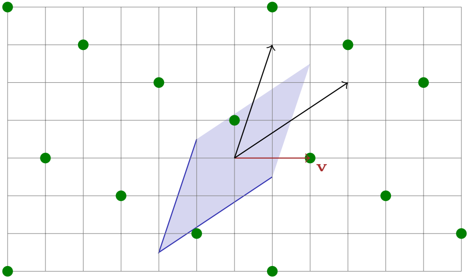

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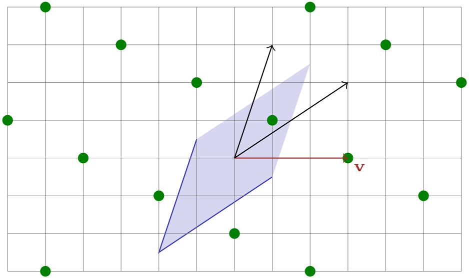

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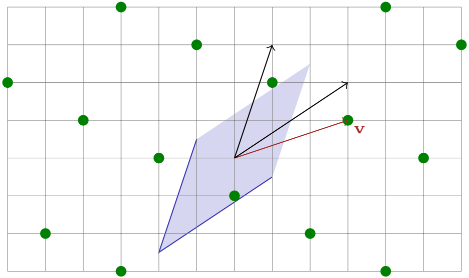

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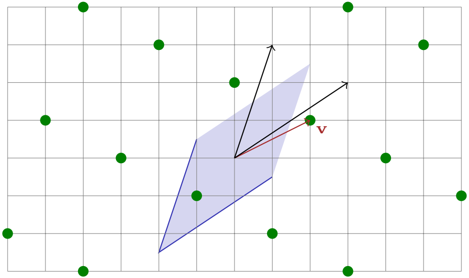

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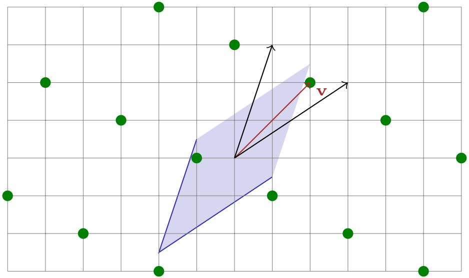

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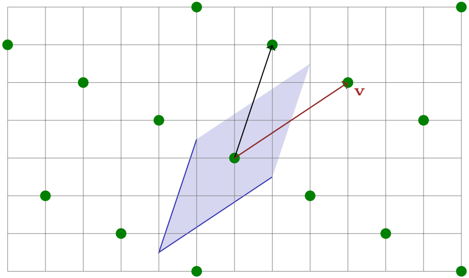

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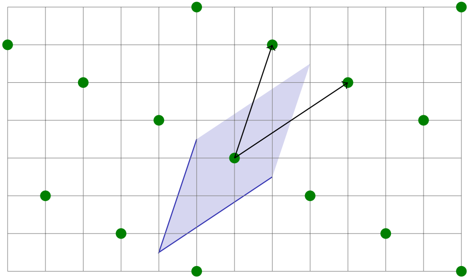

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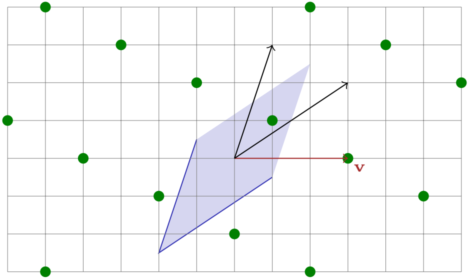
- For any basis $\mathbf{B}$ and $\mathbf{v} \in \mathbb{R}^n$, $(\mathbf{v} + \mathcal{L}) \cap \mathcal{P}(\mathbf{B}) = \{\bar{\mathbf{v}}\}$.
  Write $\bar{\mathbf{v}} = \mathbf{v} \bmod \mathbf{B}$, the "distinguished representative" of $\mathbf{v} + \mathcal{L}$.



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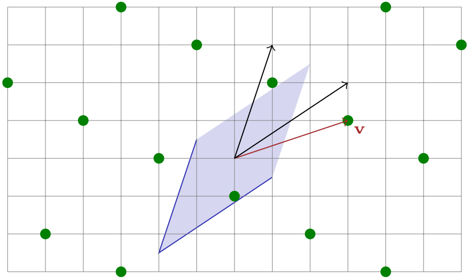
  Recall: $\mathbf{v}_1 + \mathcal{L} = \mathbf{v}_2 + \mathcal{L}$ iff $\mathbf{v}_1 - \mathbf{v}_2 \in \mathcal{L}$.
- Determinant $\det(\mathcal{L}) \stackrel{\Delta}{=} |\mathbb{Z}^n/\mathcal{L}| = |\det(\mathbf{B})| = \mathrm{vol}(\mathcal{P}(\mathbf{B}))$, any basis $\mathbf{B}$.
- For any basis $\mathbf{B}$ and $\mathbf{v} \in \mathbb{R}^n$, $(\mathbf{v} + \mathcal{L}) \cap \mathcal{P}(\mathbf{B}) = \{\bar{\mathbf{v}}\}$.
  Write $\bar{\mathbf{v}} = \mathbf{v} \bmod \mathbf{B}$, the "distinguished representative" of $\mathbf{v} + \mathcal{L}$.



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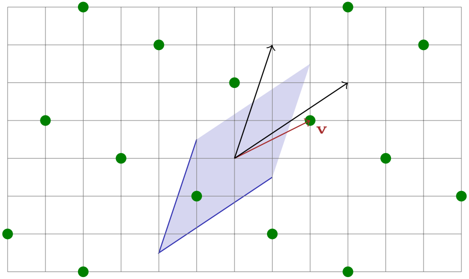
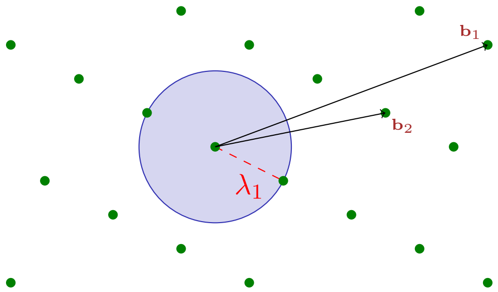


Successive Minima

- The minimum distance of $\mathcal{L}$ is

$$\lambda_1(\mathcal{L}) \stackrel{\Delta}{=} \min_{\mathbf{0} \neq \mathbf{v} \in \mathcal{L}} \|\mathbf{v}\| = \min_{\text{distinct } \mathbf{x}, \mathbf{y} \in \mathcal{L}} \|\mathbf{x} - \mathbf{y}\|.$$



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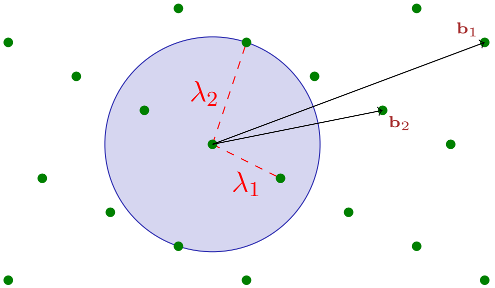
- More generally, the ith successive minimum $(i = 1, \ldots, n)$ is
$$\begin{aligned} \lambda_i(\mathcal{L}) &\triangleq \min\{r : \mathcal{L} \text{ contains } i \text{ linearly ind. vectors of length} \leq r\} \\ &= \min\{r : \dim(\operatorname{span}(\mathcal{L} \cap \mathcal{B}(r))) \geq i\}. \end{aligned}$$



Gram-Schmidt Orthogonalization and Lower Bounding λ_1

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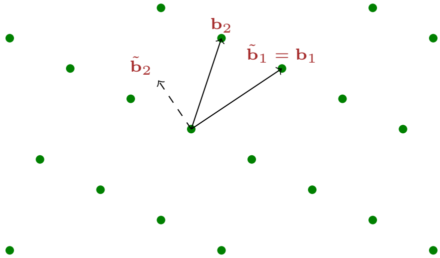

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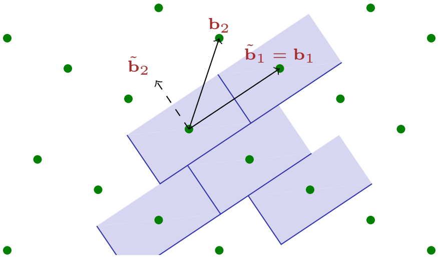

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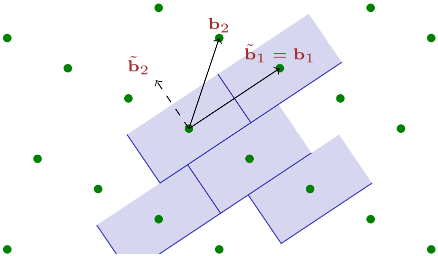

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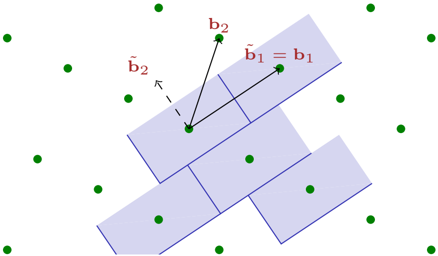

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1. Ball of radius $> \sqrt{n} \cdot \det(\mathcal{L})^{1/n}$ is convex and centrally symmetric.
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  2. Essentially all crypto schemes are based on versions of these problems.

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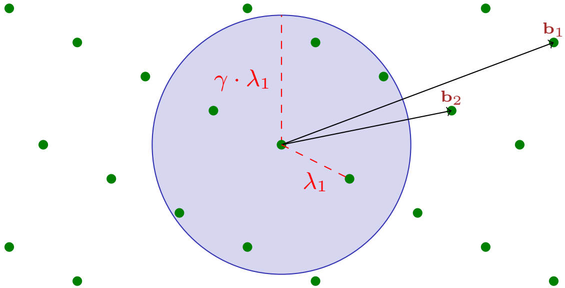
Approximation problems with factor $\gamma = \gamma(n)$:

Search: given basis $\mathbf{B}$, find nonzero $\mathbf{v} \in \mathcal{L}$ s.t. $\|\mathbf{v}\| \le \gamma \cdot \lambda_1(\mathcal{L})$.



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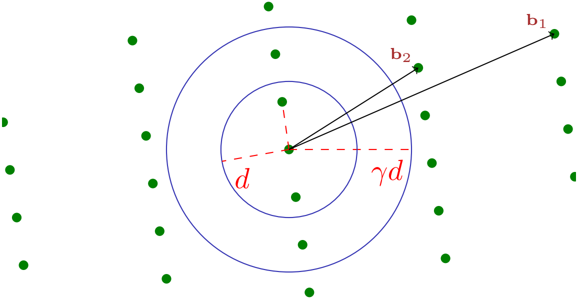

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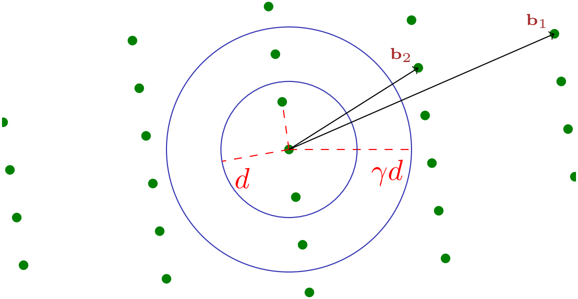

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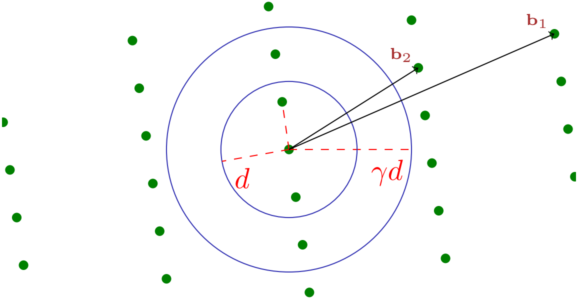
Recall: $\min_i \|\tilde{\mathbf{b}}_i\| \leq \lambda_1 \leq \sqrt{n} \cdot \det(\mathcal{L})^{1/n}$, but these are often very loose.



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- For $\gamma = 2^k$, best algorithm takes $\approx 2^{n/k}$ time [Schnorr'87,...]

  E.g., $\gamma = 2^{\sqrt{n}}$ appears to be $\approx 2^{\sqrt{n}}$-hard.

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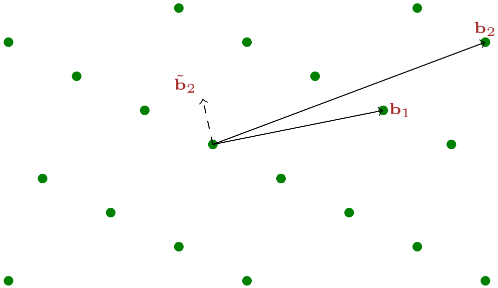

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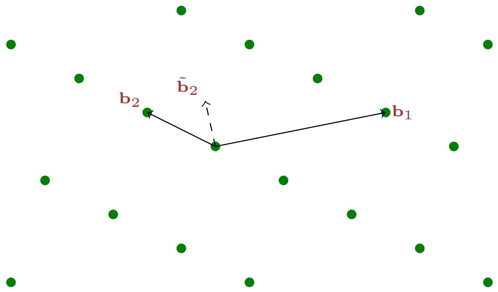

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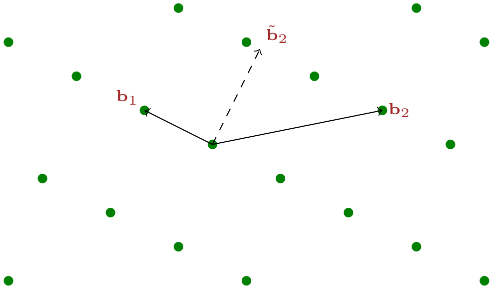

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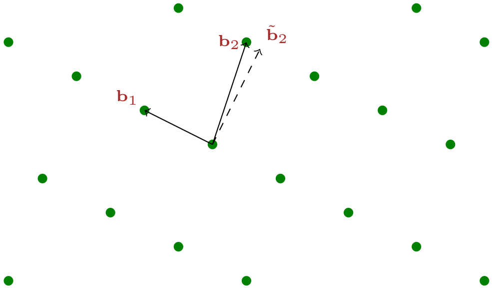

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Proof: At end, $\frac{3}{4}\|\mathbf{b}_1\|^2 \leq \|\mathbf{b}_2\|^2 \leq \|\tilde{\mathbf{b}}_2\|^2 + \frac{1}{4}\|\mathbf{b}_1\|^2$.

Claim 2: Algorithm terminates after poly($|\mathbf{B}|$) many iterations.

Proof: Define $\Phi(\mathbf{B}) = \|\tilde{\mathbf{b}}_1\|^2 \cdot \|\tilde{\mathbf{b}}_2\| = \|\mathbf{b}_1\| \cdot \det(\mathcal{L})$.

When we swap, Φ decreases by $> \frac{\sqrt{3}}{2}$ factor.

It starts as $2^{\text{poly}(|\mathbf{B}|)}$ and cannot go below 1.

An Algorithm for $\mathrm{SVP}_{2^{(n-1)/2}}$ [LLL'82]

- Key idea: manipulate basis to ensure $\|\tilde{\mathbf{b}}_{i+1}\|^2 \geq \frac{1}{2}\|\tilde{\mathbf{b}}_i\|^2$, for all i.

  This implies $\|\mathbf{b}_1\| \leq 2^{(n-1)/2} \cdot \min_i \|\tilde{\mathbf{b}}_i\| \leq 2^{(n-1)/2} \cdot \lambda_1(\mathcal{L})$.

In two dimensions: given basis $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2)$,

1. Let $\mathbf{b}_2 \leftarrow \mathbf{b}_2 - c \cdot \mathbf{b}_1$ for the $c \in \mathbb{Z}$ s.t. $\mathbf{b}_2 \in \tilde{\mathbf{b}}_2 + [-\frac{1}{2}, \frac{1}{2}) \cdot \mathbf{b}_1$.
2. If $\|\mathbf{b}_2\|^2 < \frac{3}{4}\|\mathbf{b}_1\|^2$, swap $\mathbf{b}_1 \leftrightarrow \mathbf{b}_2$ and loop. Else end.

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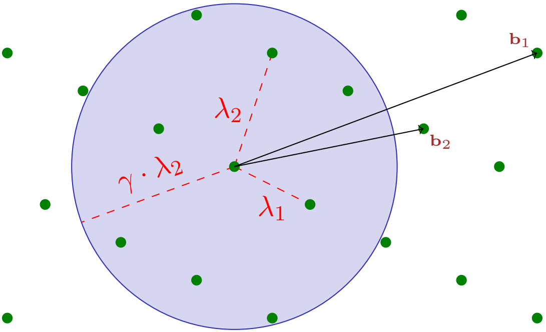
It starts as $2^{\mathrm{poly}(|\mathbf{B}|)}$ and cannot go below 1.

LLL in n dimensions: do similar loop on all adjacent pairs $\mathbf{b}_i, \mathbf{b}_{i+1}$.

Related: Shortest Independent Vectors Problem (SIVP_γ)

- Given basis $\mathbf{B}$, <u>find</u> lin. ind. $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathcal{L}$ s.t. $\|\mathbf{v}_i\| \le \gamma \cdot \lambda_n(\mathcal{L})$.



Related: Shortest Independent Vectors Problem (SIVP$_\gamma$)

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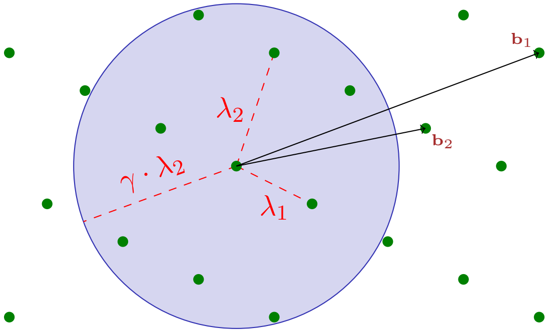

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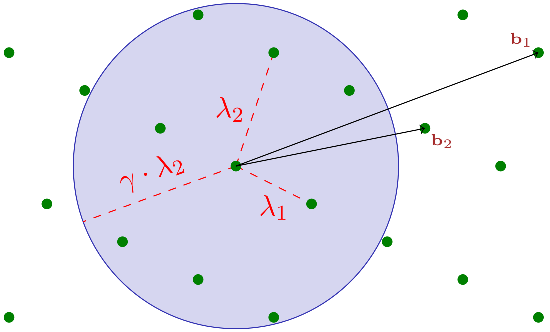
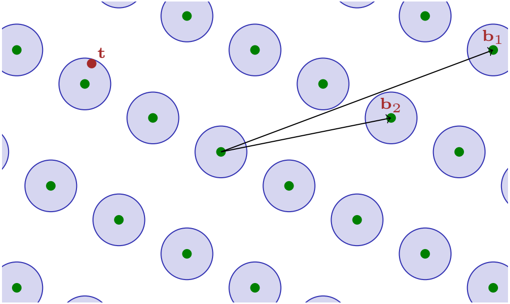
- LLL algorithm also solves $\mathsf{SIVP}_{2^{(n-1)/2}}$.
- We know $\mathsf{GapSVP}_\gamma \leq \mathsf{SIVP}_\gamma$, but the reverse direction is open!



Bounded-Distance Decoding (BDD)

Search: given basis $\mathbf{B}$, point $\mathbf{t}$, and real $d < \lambda_1/2$ s.t. $\mathrm{dist}(\mathbf{t}, \mathcal{L}) \leq d$, find the (unique) $\mathbf{v} \in \mathcal{L}$ closest to $\mathbf{t}$.



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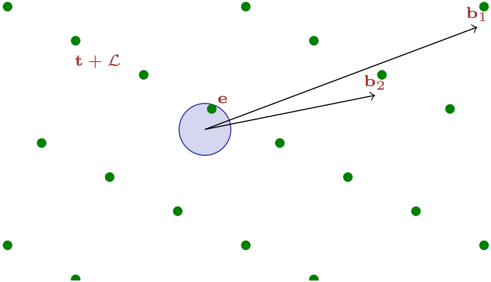

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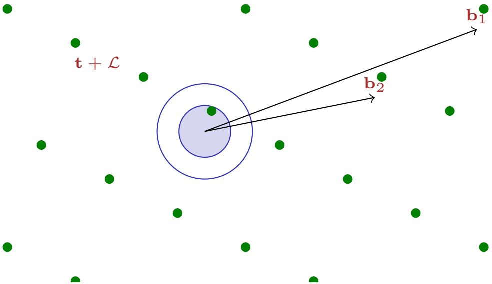

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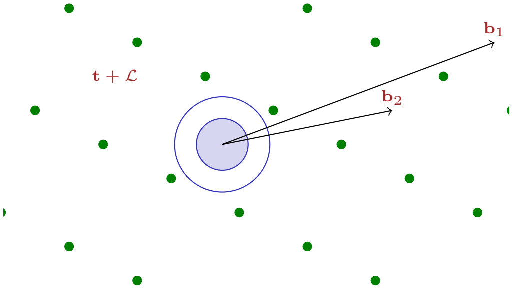
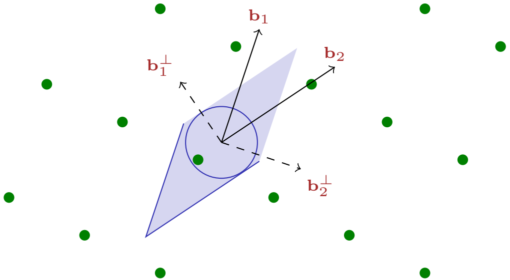


Algorithms for BDD [Babai'86]

"Round off:" Using a "good" basis $\mathbf{B}$, output $\mathbf{e} = \mathbf{t} \bmod \mathbf{B}$.

Works if Ball$(d) \subseteq \mathcal{P}(\mathbf{B})$: radius $d = \min_i \|\mathbf{b}_i^\perp\|/2$.



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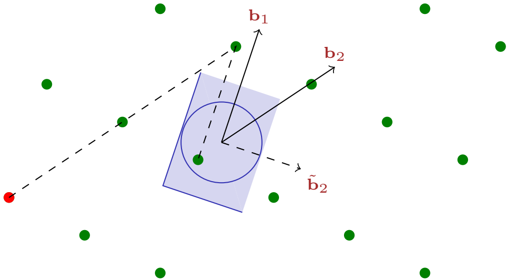
"Nearest plane:" Output $\mathbf{e} = \mathbf{t} \bmod \tilde{\mathbf{B}}$. Proceeds iteratively.

Works if $\text{Ball}(d) \subseteq \mathcal{P}(\tilde{\mathbf{B}})$: radius $d = \min_i \|\tilde{\mathbf{b}}_i\|/2$.



Wrapping Up

- Now you know (almost) everything you need to know about lattices (to do cryptography, at least).
- We've covered a lot: do the exercises to reinforce your understanding!
- Tomorrow: the cryptographic problems SIS and LWE (as SVP and BDD variants), and some basic applications.