ARC Fellowship Proposal: Towards the KLS Conjecture for Convex Bodies

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Abstract

We propose developing new tools and approaches towards resolving the conjecture posed in [KLS95], which posits that the isoperimetric coefficient of a convex body K should be proportional to $1/\lambda(K)$, where $\lambda(K)$ denotes the top eigen value of the covariance matrix of K. This research would extend the work done in [CDV09], as well as the research pursued by the applicant while visiting professor Klartag over the summer in Tel Aviv.

Concentration of Measure in High Dimensions

A remarkable phenomena in high dimensional geometry is that of concentration of mass, i.e. that the mass of a body becomes more and more concentrated towards its boundary as dimension increases. This is best illustrated by the classical isoperimetric inequality in $\mathbb{R}^n$, which states that among all bodies of same volume the euclidean ball minimizes the surface area to volume ratio. More precisely, letting $S \subseteq \mathbb{R}^n$ be a measurable subset of $\mathbb{R}^n$, where $|S|$ denotes its volume (Lebesgue measure) and $|\partial S|$ is the $n-1$ dimensional volume of the boundary of S, we have that

$$\frac{|\partial S|}{|S|} \geq n \left(\frac{v_n}{|S|}\right)^{\frac{1}{n}}$$

where v_n denotes volume of the n dimensional euclidean ball. Here we note that the ratio $(|S|/v_n)^{\frac{1}{n}}$ is simply the radius of the ball whose volume is $|S|$. For a body S of volume 1, the above tells us that $|\partial S|/|S| \geq \sqrt{\frac{n}{2\pi}}$ (roughly). Consequently, if we take an ϵ extension of S (for small ϵ), i.e. we take all points at distance ϵ from S, we get that the volume of the extension is at least $(1 + \epsilon\sqrt{\frac{n}{2\pi}})$ larger than that of S. Hence to double the volume of S (i.e. to go from volume 1 to 2), it suffices (roughly) to take $\sqrt{2\pi/n}$ extension of S. Another way to interpret this is that almost all the volume of S is within distance $\sqrt{2\pi/n}$ of ∂S!

Isoperimetric Inequalities for Convex Bodies

For this research, we study a different but very related notion of concentration. Our domain of study will be convex bodies where our main interest lies in lower bounding the size of the "internal boundaries" of their subsets. Formally, let $K \subseteq \mathbb{R}^n$ be a convex body, and let $A \subseteq K$ be a measurable subset of K. We define the internal boundary of A with respect to K:

$$\partial_K A = \partial A \setminus \partial K$$

E.g. $\partial_K A$ is the boundary of A that lies in the interior of K. The main question we wish to answer is the following: what is the largest number $\psi(K)$ such that $\forall A \subset K$, A measurable, we have that

$$|\partial_K A| \geq \psi(K) \min\{|A|, |K \setminus A|\}$$

The quantity $\psi(K)$ is called the isoperimetric coefficient of K. We now describe some of the known results in the area. For a convex body $K \subset \mathbb{R}^n$, we define its covariance matrix as follows

$$\mathrm{cov}(K)_{ij} = \frac{1}{|K|}\int_K x_i x_j dx - \left(\frac{1}{|K|}\int_K x_i dx\right)\left(\frac{1}{|K|}\int_K x_j dx\right) \qquad \text{for } 1 \leq i,j \leq n.$$

Let $\lambda(K)$ denote largest eigen value of $\mathrm{cov}(K)$ and let $M_2(K) = \mathrm{trace}(\mathrm{cov}(K))$. In [KLS95], it was shown that

$$\psi(K) \geq \frac{c_1}{\sqrt{M_2(K)}} \quad \text{and} \quad \psi(K) \leq \frac{c_2}{\lambda(K)}$$

where $c_1, c_2 > 0$ are absolute constants. There have been improvements to the lower bound, which are most easily stated when $\mathrm{cov}(K) = I$, i.e. the identity matrix. In this case, the above bound tells us that $\psi(K) = \Omega(1/\sqrt{n})$. The best lower bound in the general case is due [Bob07] which gives $\psi(K) = \Omega(1/n^{.46})$, and in the case where K is absolutely symmetric (i.e. symmetric about any coordinate axis) [Kla08] showed that $\psi(K) = \Omega(1/\log n)$.

The remarkable conjecture made in [KLS95], and possibly the biggest open problem in asymptotic convex geometry, posits that the correct answer should be

$$\psi(K) = \Theta(\lambda(K))$$

Note that when $\mathrm{cov}(K)$ is the identity matrix, the above implies that $\psi(K) = O(1)$, and hence is dimension independent!

Tools and Approaches

The most successful tool for proving isoperimetric inequalities to date is the method of localization developed by Lovasz and Simonovits in [LS93]. The power of localization is that it reduces n-dimensional inequalities to 1-dimensional ones. Localization achieves this by a method of bisections, where it uses a sequence of halfspace cuts to reduce an inequality over a convex body K to one over very thin / almost 1-dimensional pieces of the K which come from the halfspace cut partition of K. The main caveat of localization is that if any of these near 1-dimensional pieces is "bad" for the inequality we wish to prove, the whole method breaks down.

Our contribution so far has been to show that in some cases, one can ignore a small number of partition pieces and prove the isoperimetric inequality using the remaining "good" pieces. In [CDV09], we used this idea to prove that star-shaped sets satisfy non-trivial isoperimetric inequalities, something which would have been impossible to do with standard localization.

The main focus of our research will be to develop new ways to analyze an entire halfspace cut partition of a convex body into "thin" pieces. We believe that any non-trivial advances in our understanding of these partitions will lead to significant improvements in current isoperimetric inequalities.

Motivation

An important motivation for this research is that isoperimetric inequalities have played a major role in many geometric algorithms, most notably in the analysis of random walk methods for optimizing, integrating, and sampling logconcave functions over convex bodies [LV06]. The above inequalities are the main tools used to prove that certain geometric random walks are rapidly mixing, where in essence they help show that the random walk cannot get "stuck" in small subsets of K. Lastly, the isoperimetric coefficient is one of the most important and difficult geometric quantities to analyze, and we believe the tools developed to study it will have broad applications in geometry, probability theory, algorithms, and other areas.

References

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