# CS 4644 / 7643-A

# DEEP LEARNING: LECTURE 3

## DANFEI XU

- Linear Classifier (cont.)
- SVM / Hinge Loss
- Softmax Classifier and Cross-Entropy Loss
- Gradient Descent

# MISC

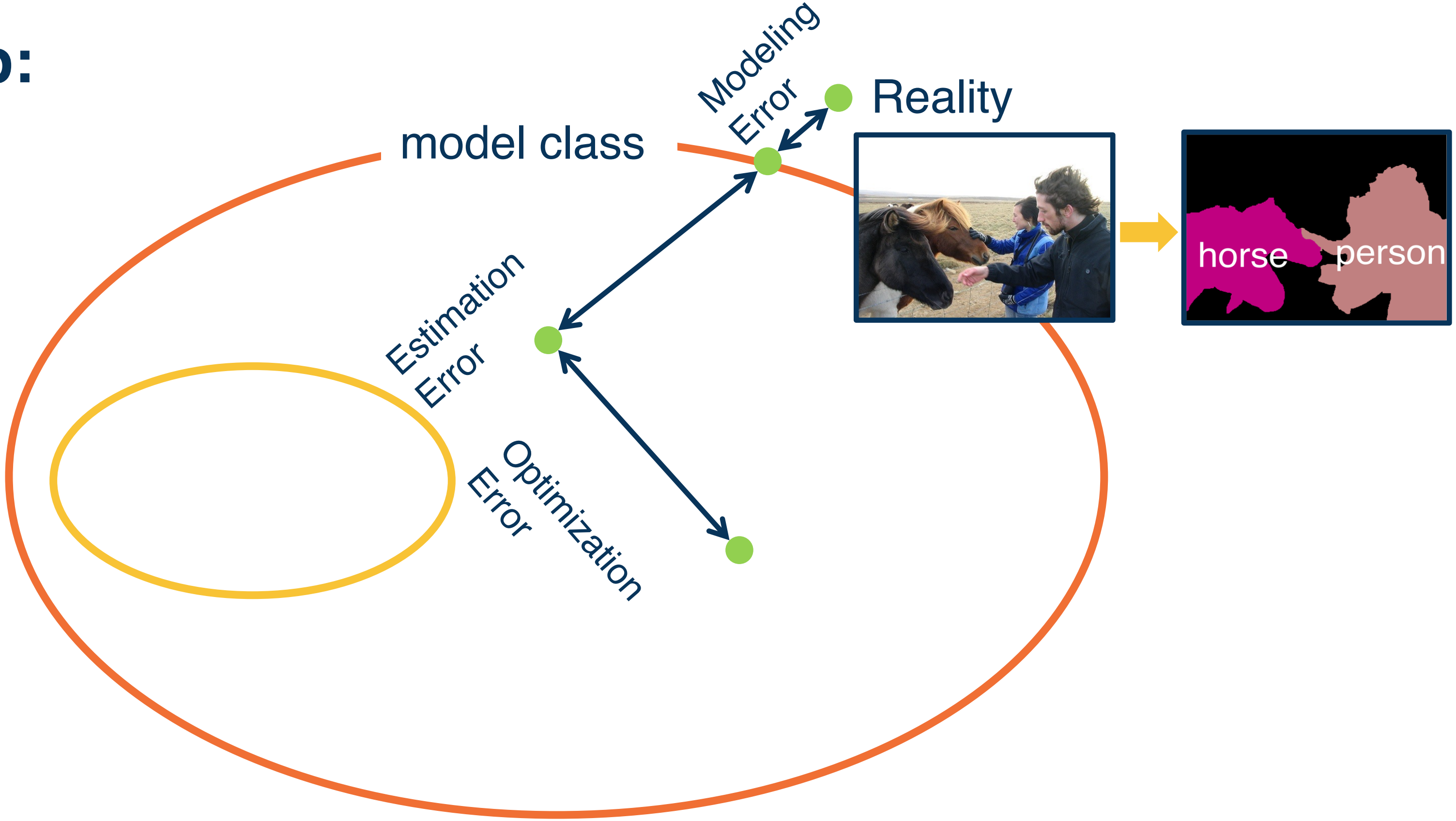
- Make sure you know how to use Google Colab
- PS1 release 08/29
- Use Piazza!
- Start to find your project team!
- Shared sample project report in @6 on Piazza.

# Recap:

| Supervised Learning | Unsupervised Learning | Reinforcement Learning |
|---|---|---|
| • Train Input: $\{X, Y\}$<br>• Learning output: $f : X \rightarrow Y$, e.g. $P(y\|x)$ | • Input: $\{X\}$<br>• Learning output: $P(x)$<br>• Example: Clustering, density estimation, etc. | • Supervision in form of **reward**<br>• No supervision on what action to take |

**Very often combined**, sometimes within the same model!

# Recap:

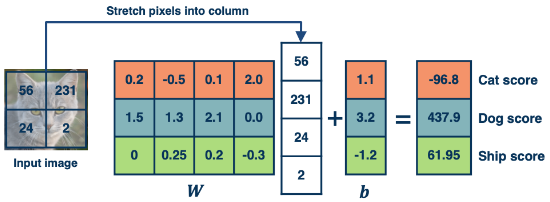


model class
Modeling Error
Reality
Estimation Error
Optimization Error
horse
person

# Recap:

## Algebraic Viewpoint

$$f(x, W) = Wx$$

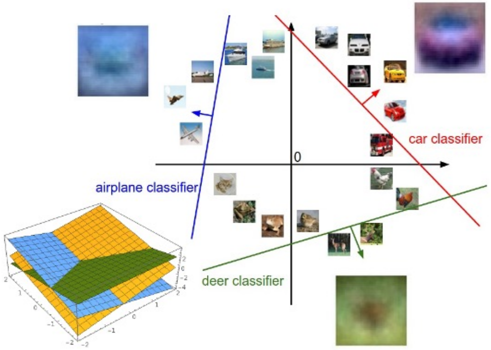



## Visual Viewpoint

One template per class


## Geometric Viewpoint

Hyperplanes cutting up space




*Adapted from from CS 231n slides*

# This time:

$$f(x, W) = Wx$$

| | | | |
|---|---|---|---|
| airplane | -3.45 | -0.51 | 3.42 |
| automobile | -8.87 | **6.04** | 4.64 |
| bird | 0.09 | 5.31 | 2.65 |
| cat | **2.9** | -4.22 | 5.1 |
| deer | 4.48 | -4.19 | 2.64 |
| dog | 8.02 | 3.58 | 5.55 |
| frog | 3.78 | 4.49 | **-4.34** |
| horse | 1.06 | -4.37 | -1.5 |
| ship | -0.36 | -2.09 | -4.79 |
| truck | -0.72 | -2.93 | 6.14 |

1.Define a **loss function** that quantifies our unhappiness with the scores across the training data.

2. Come up with a way of efficiently finding the parameters that minimize the loss function. **(optimization)**

Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x,W)=Wx$ are:

| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
| car | 5.1 | **4.9** | 2.5 |
| frog | -1.7 | 2.0 | **-3.1** |
| | High Loss | Low Loss | High Loss |

A **loss function** that tells how good the current classifier is

Given a dataset of examples:

$$\{(x_i, y_i)\}_{i=1}^{N}$$

Where $x_i$ is image and
$y_i$ is (integer) label

Loss over the dataset is a sum of loss over examples:

$$L = \frac{1}{N}\sum L(f(x_i, W), y_i)$$

*Adapted from from CS 231n slides*

# SVM Loss Example

# Multiclass SVM loss:

Given an example $(x_i, y_i)$
where $x_i$ is the image and
where $y_i$ is the (integer) label,

and using the shorthand for the
scores vector: $s_i = f(x_i, W)$

**Notation:** $s_{y_i}$ is the **score** given by the classifier for the correct label class of the i-th example ($y_i$)

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

$$= \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

# Multiclass SVM loss:

Given an example $(x_i, y_i)$ where $x_i$ is the image and where $y_i$ is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

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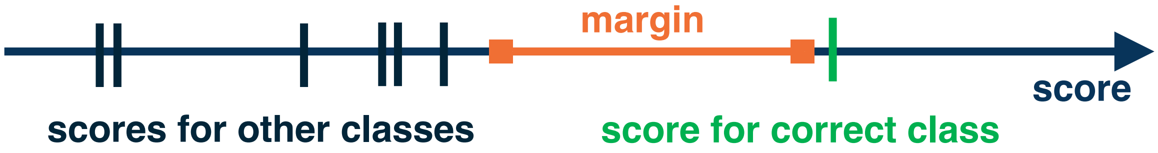
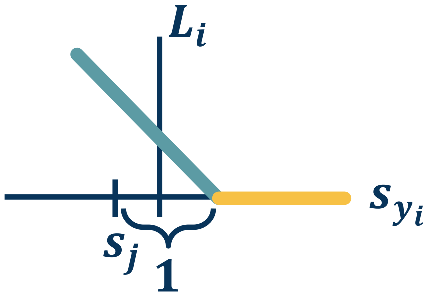
$$L_i = \sum_{j \neq y_i} \begin{cases} \boxed{0 \quad \text{if } s_{y_i} \geq s_j + 1} \\ \boxed{s_j - s_{y_i} + 1 \quad \text{otherwise}} \end{cases}$$

$$= \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Loss = 0:




"Hinge Loss"



## Multiclass SVM loss:

Given an example $(x_i, y_i)$ where $x_i$ is the image and where $y_i$ is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)}$$

= max(0, 5.1 - 3.2 + 1) + max(0, -1.7 - 3.2 + 1)
= max(0, 2.9) + max(0, -3.9)
= 2.9 + 0
= 2.9

Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x,W)=Wx$ are:



| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
| car | 5.1 | **4.9** | 2.5 |
| frog | -1.7 | 2.0 | **-3.1** |
| **Losses:** | **2.9** | | |

*Adapted from from CS 231n slides*

## SVM Loss Example

# Multiclass SVM loss:

Given an example $(x_i, y_i)$ where $x_i$ is the image and where $y_i$ is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)}$$

= max(0, 1.3 - 4.9 + 1) + max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x, W) = Wx$ are:



| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
| car | 5.1 | **4.9** | 2.5 |
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| **Losses:** | 2.9 | **0.0** | |

*Adapted from from CS 231n slides*

## SVM Loss Example

## Multiclass SVM loss:

Given an example $(x_i, y_i)$ where $x_i$ is the image and where $y_i$ is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

L = (2.9 + 0 + 12.9)/3
= **5.27**

Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x,W) = Wx$ are:



| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
| car | 5.1 | **4.9** | 2.5 |
| frog | -1.7 | 2.0 | **-3.1** |
| Losses: | 2.9 | 0 | 12.9 |

*Adapted from from CS 231n slides*

# SVM Loss Example

## Multiclass SVM loss:

Suppose: 3 training examples, 3 classes.
With some $\boldsymbol{W}$ the scores $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{W})=\boldsymbol{W}\boldsymbol{x}$ are:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: What happens to loss if car image scores change a bit (e.g., $\pm$ 0.1)?

No change for small values



| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
| car | 5.1 | **4.9** | 2.5 |
| frog | -1.7 | 2.0 | **-3.1** |

*Adapted from from CS 231n slides*

## Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: What is min/max of loss value?

[0,inf]

Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x,W)=Wx$ are:



| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
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*Adapted from from CS 231n slides*

## Multiclass SVM loss:

Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x,W)=Wx$ are:

$$L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: At initialization W is close to 0 so all s ≈ 0. What is the loss?

num_class - 1



| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
| car | 5.1 | **4.9** | 2.5 |
| frog | -1.7 | 2.0 | **-3.1** |

*Adapted from from CS 231n slides*

# SVM Loss Example

## Multiclass SVM loss:

Suppose: 3 training examples, 3 classes.
With some $W$ the scores $f(x,W)=Wx$ are:

$$L_i = \frac{1}{C}\sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$$

Q: What if we used mean instead of sum?

No difference
Scaling by constant

| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
| car | 5.1 | **4.9** | 2.5 |
| frog | -1.7 | 2.0 | **-3.1** |

*Adapted from from CS 231n slides*

# Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

```
def hinge_loss_vec(x, y, W):
  """
  x (d): input example vectors
  y (int): class label
  W (C x d): weight matrix
  """
  scores = W.dot(x)   # calculate raw scores
  margins = np.maximum(0, scores - scores[y] + 1) # calculate margins s_j - s_{yi} + 1
  margins[y] = 0 # exclude yi from the loss sum
  loss_i = np.sum(margins). # sum across all j (classes)
  return loss_i
```

*Adapted from from CS 231n slides*

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

E.g. Suppose that we found a W such that L = 0.

Q: Is this W unique?

Let's look at an example

*Adapted from from CS 231n slides*

# Multiclass SVM loss:

Suppose: 3 training examples, 3 classes.
With some $\boldsymbol{W}$ the scores $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{W})=\boldsymbol{W}\boldsymbol{x}$ are:

| | | | |
|---|---|---|---|
| cat | **3.2** | 1.3 | 2.2 |
| car | 5.1 | **4.9** | 2.5 |
| frog | -1.7 | 2.0 | **-3.1** |

Before:
= max(0, 1.3 - 4.9 + 1)
+max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

With W **twice as large:**
= max(0, 2.6 - 9.8 + 1)
+max(0, 4.0 - 9.8 + 1)
= max(0, -6.2) + max(0, -4.8)
= 0 + 0
= 0

*Adapted from from CS 231n slides*

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

E.g. Suppose that we found a W such that L = 0.

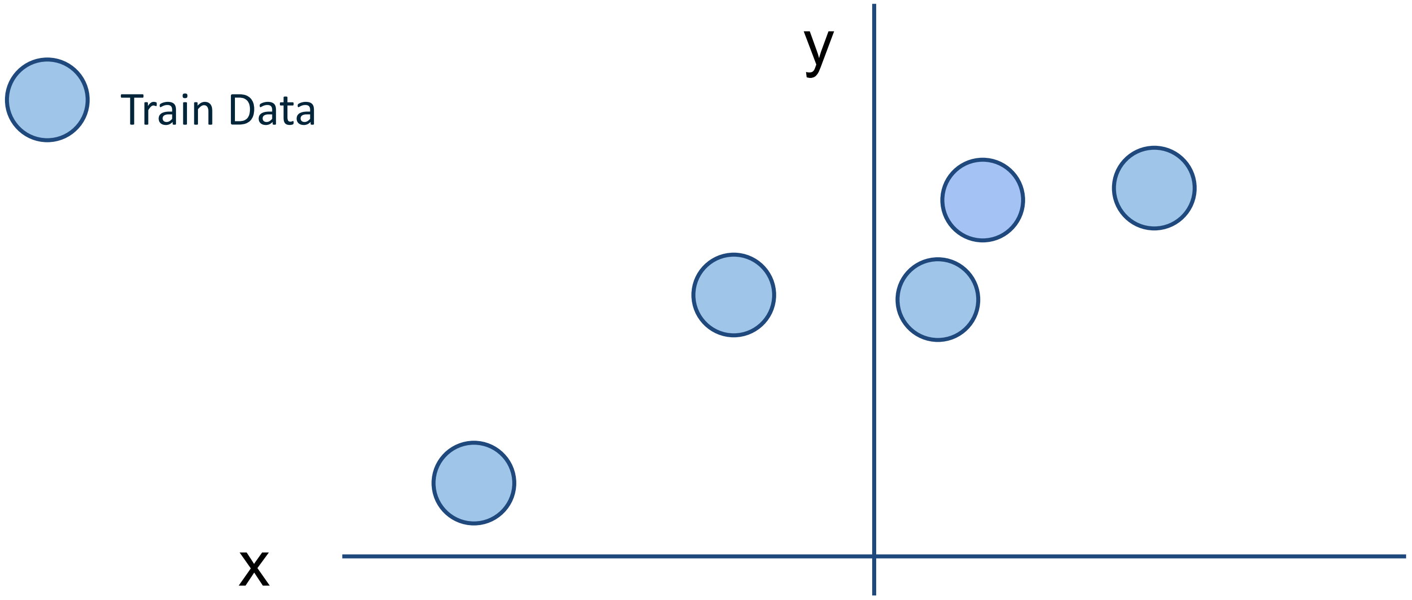
Q: Is this W unique?

No, 2W also has L=0

How do we choose between W, 2W, and 1e+7W?

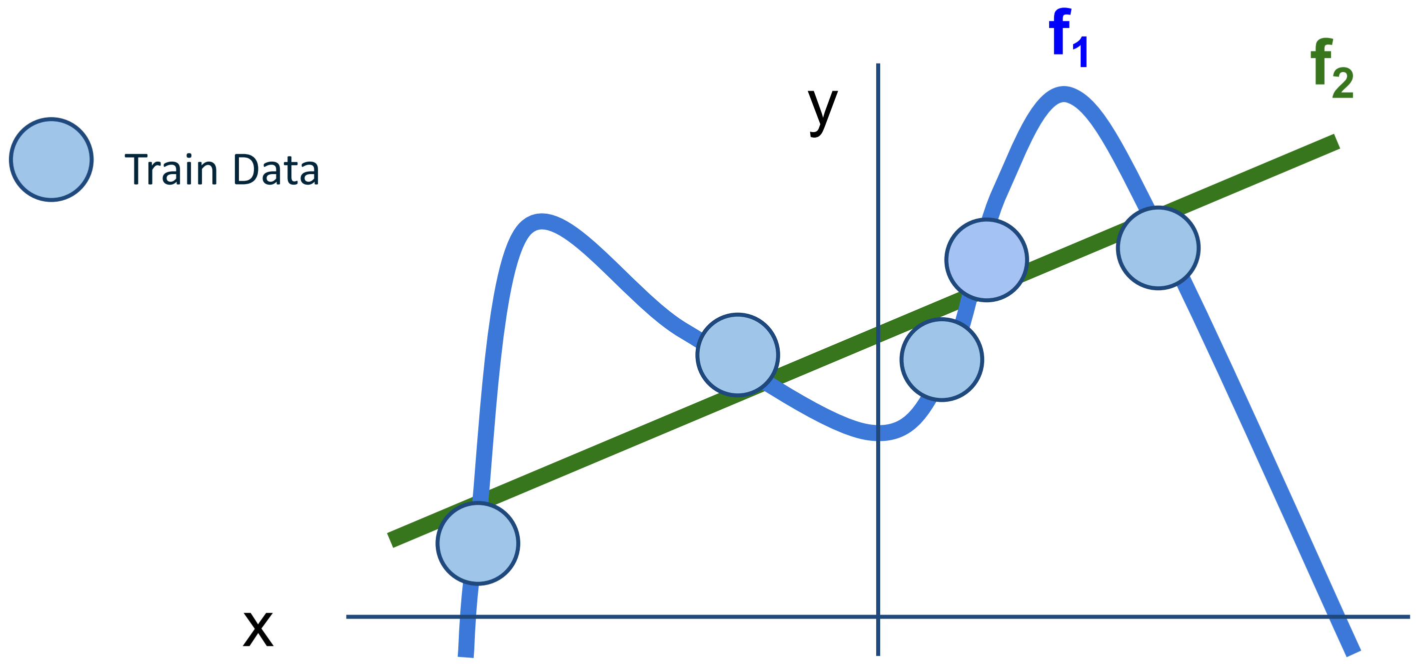
*Adapted from from CS 231n slides*

# Regularization intuition: fitting a polynomial function



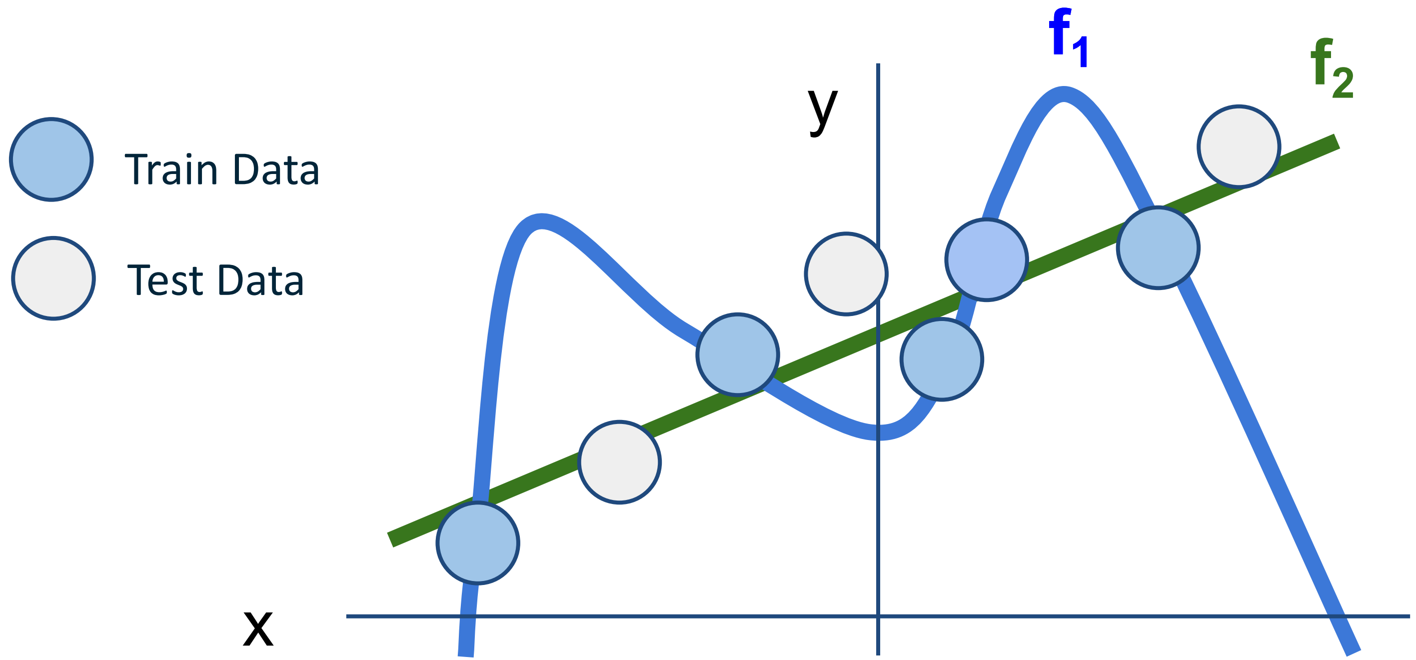

*Adapted from from CS 231n slides*

# Regularization intuition: fitting a polynomial function




*Adapted from from CS 231n slides*

# Regularization intuition: fitting a polynomial function




Regularization balances the simplicity of the function and loss, so we don't overfit to the noises in the data

*Adapted from from CS 231n slides*

# Regularization

$$L(W) = \underbrace{\frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)}_{} + \underbrace{\lambda R(W)}_{}$$

**Data loss**: Model predictions should match training data

**Regularization**: Prevent the model from doing *too* well on training data

*Adapted from from CS 231n slides*

Reference: https://icml.cc/Conferences/2004/proceedings/papers/354.pdf

# Regularization

$\lambda$ = regularization strength (hyperparameter)

$$\underbrace{L(W) = \frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)}_{\text{Data loss}} + \underbrace{\lambda R(W)}_{\text{Regularization}}$$

**Data loss**: Model predictions should match training data

**Regularization**: Prevent the model from doing *too* well on training data

**Simple examples**

<u>L2 regularization:</u> $R(W) = \sum_k \sum_l W_{k,l}^2$

L1 regularization: $R(W) = \sum_k \sum_l |W_{k,l}|$

Elastic net (L1 + L2): $R(W) = \sum_k \sum_l \beta W_{k,l}^2 + |W_{k,l}|$

**More complex (DNN-specific)**:

Dropout

Batch/layer normalization

Stochastic depth, fractional pooling, etc

# Regularization: Implement a simple L2 regularizer

$$L(W) = \frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i) + \lambda R(W)$$

```
def l2_regularized_hinge_loss(x, y, W, reg_coeff):
  data_loss = 0
  # calculate dataset loss
  for i in range(x.shape[0]):
    data_loss += hinge_loss_vec(x[i], y[i], W)

  # calculate weight regularization loss
  reg_loss = np.sum(np.square(W)) * reg_coeff

  return data_loss + reg_loss
```

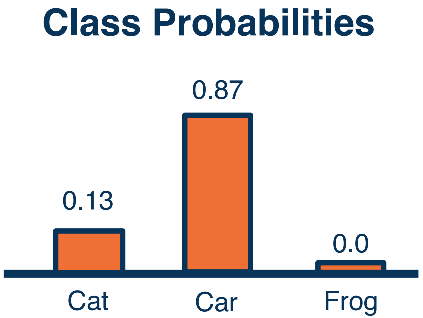
# What if we want probabilities?

We need a different classifier!*

| | |
|---|---|
| cat | **3.2** |
| car | 5.1 |
| frog | -1.7 |

**Raw class scores**

→ ? →




*Technically we can get probability from SVM classifiers too, see [Platt scaling]

# **Softmax Classifier** (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; \theta)}$$

Probabilities must be >= 0

$$\boxed{p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}}$$

**Softmax Function**

Probabilities must sum to 1

| | | | | | |
|---|---|---|---|---|---|
| cat | **3.2** | | **24.5** | | **0.13** |
| car | 5.1 | exp → | 164.0 | normalize → | 0.87 |
| frog | -1.7 | | 0.18 | | 0.00 |
| | **Unnormalized log-probabilities / logits** | | **Unnormalized probabilities** | | **Probabilities** |

How do we compute the loss?

*Adapted from from CS 231n slides*

# **Softmax Classifier** (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; \theta)}$$

$$\boxed{p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}}$$

**Softmax Function**

We maximize the probability of $p_\theta(y_i|x_i)$!

| | Unnormalized log-probabilities / logits | | Predicted Probs (softmax) |
|---|---|---|---|
| cat | **3.2** | softmax → | **0.13** |
| car | 5.1 | | 0.87 |
| frog | -1.7 | | 0.00 |

Finding a set of weights $\theta$ that maximizes the probability of correct prediction: $\underset{\theta}{\text{argmax}} \prod p_\theta(y_i|x_i)$

This is equivalent to:

$$\underset{\theta}{\text{argmax}} \sum \ln p_\theta(y_i|x_i)$$

$$L_i = -\ln p_\theta(y_i|x_i) = -\ln\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) = -\ln(0.13)$$

**1. Maximum Likelihood Estimation (MLE):** Choose weights to maximize the likelihood of observed data. In this case, the loss function is the **Negative Log-Likelihood (NLL)**.

# **Softmax Classifier** (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; \theta)}$$

$$\boxed{p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}}$$

**Softmax Function**

We maximize the probability of $p_\theta(y_i|x_i)$!

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|---|---|---|---|
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| car | 5.1 | | 0.87 |
| frog | -1.7 | | 0.00 |

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This is equivalent to:

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$$L_i = -\ln p_\theta(y_i|x_i) = -\ln\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) = -\ln(0.13)$$

**Negative Log Likelihood (NLL)**

# **Softmax Classifier** (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; \theta)}$$

$$\boxed{p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}}$$

**Softmax Function**

2. Information theory view

| | | | | |
|---|---|---|---|---|
| cat | 3.2 | 0.13 | | 1.00 |
| car | 5.1 | 0.87 | | 0.00 |
| frog | -1.7 | 0.00 | | 0.00 |

softmax

maximize agreement

**Unnormalized log-probabilities / logits**

**Predicted Probs (softmax)**

**Correct probs**

*Adapted from from CS 231n slides*

# **Softmax Classifier** (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; \theta)$$

$$p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}$$

**Softmax Function**

2. Information theory view

| | Unnormalized log-probabilities / logits | | Predicted Probs (softmax) | | Correct probs |
|---|---|---|---|---|---|
| cat | **3.2** | softmax → | **0.13** | → maximize agreement ← | **1.00** |
| car | 5.1 | | 0.87 | | 0.00 |
| frog | -1.7 | | 0.00 | | 0.00 |

***Cross Entropy:*** $H(p, q) = -\sum p(x) \ln q(x)$

***Cross Entropy Loss -> NLL***

$$H_i(p, p_\theta) = -\sum_{y \in Y} p(y|x_i) \ln p_\theta(y|x_i)$$

$$= -\ln p_\theta(y_i|x_i)$$

$$L = \sum H_i(p, p_\theta) = -\sum \ln p_\theta(y_i|x_i) \equiv NLL$$

*Adapted from from CS 231n slides*

# **Softmax Classifier** (Multinomial Logistic Regression)

NLL and CrossEntropy are different loss functions in PyTorch!

## CROSSENTROPYLOSS

CLASS `torch.nn.CrossEntropyLoss(weight=None, size_average=None, ignore_index=- 100, reduce=None, reduction='mean', label_smoothing=0.0)` [SOURCE]

Expects unformalized logits as input (the function will apply softmax & log on top)

## NLLLOSS

CLASS `torch.nn.NLLLoss(weight=None, size_average=None, ignore_index=- 100, reduce=None, reduction='mean')` [SOURCE]

Expects log probabilities as input (do softmax yourself!)

# **Softmax Classifier** (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{\boldsymbol{s = f(x_i; \theta)}}$$

$$\boxed{p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}}$$

**Softmax Function**

**Cross-entropy loss:**

$$L_i = -\log(p_\theta(y_i | x_i))$$

Q: What is the min/max of possible loss L_i?

Infimum is 0, max is unbounded (inf)

*Adapted from from CS 231n slides*

# **Softmax Classifier** (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; \theta)}$$

$$\boxed{p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}}$$

**Softmax Function**

**Cross-entropy loss:**

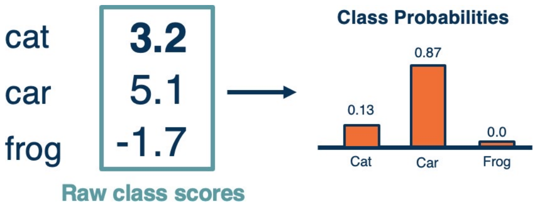
$$L_i = -\log(p_\theta(y_i | x_i))$$

Q: At initialization all s will be approximately equal; what is the loss?

-log(1/C), e.g. -log(1/3)=log(3) ≈ 1.1

*Adapted from from CS 231n slides*

# Q: Why softmax?




Why this?

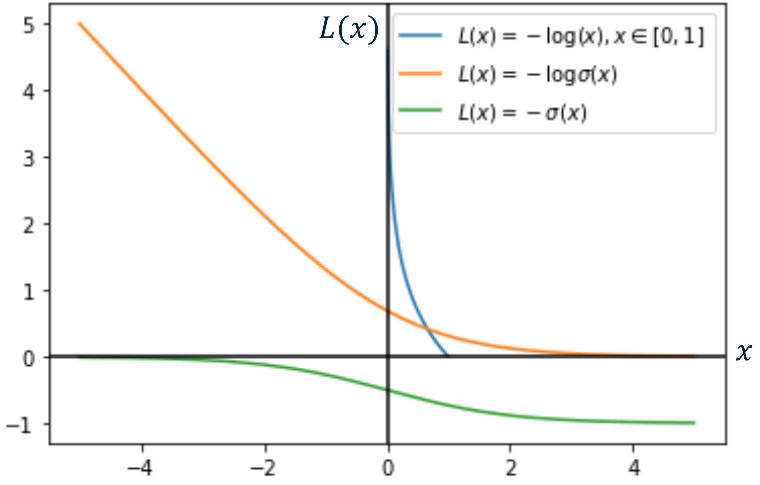
$$\boxed{p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}}$$

Use logistic function as example. Same as general softmax but for binary classification

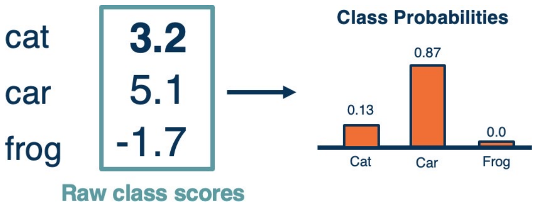
$$\sigma(x) = \frac{e^x}{1 + e^x}$$

Consider the following three basis for NLL:

1. Squash and clip network value (x) to (0, 1]
2. (Negative) logistic function
3. NLL with logistic function



# Q: Why softmax?




Why this?

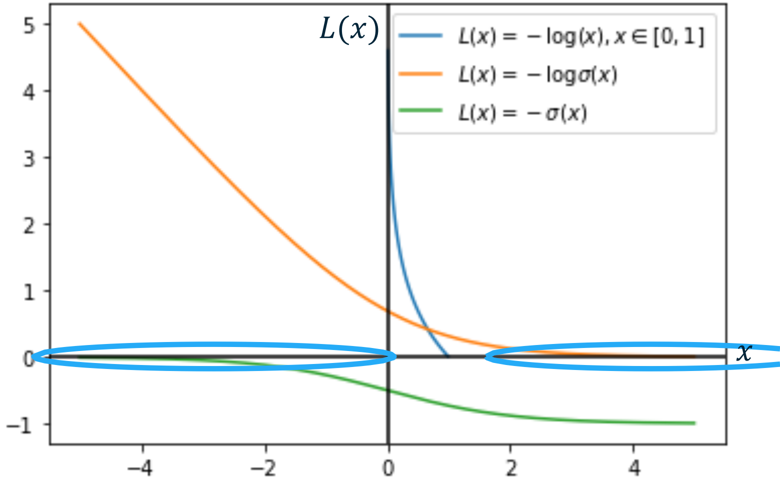
$$p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}$$

Use logistic function as example. Same as general softmax but for binary classification

$$\sigma(x) = \frac{e^x}{1 + e^x}$$

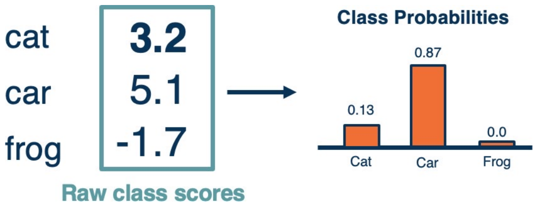
Consider the following three basis for NLL:

1. Squash and clip network value (x) to (0, 1]
2. (Negative) logistic function
3. NLL with logistic function




1. Squash & clip: no loss, no learning!

# Q: Why softmax?




Why this?

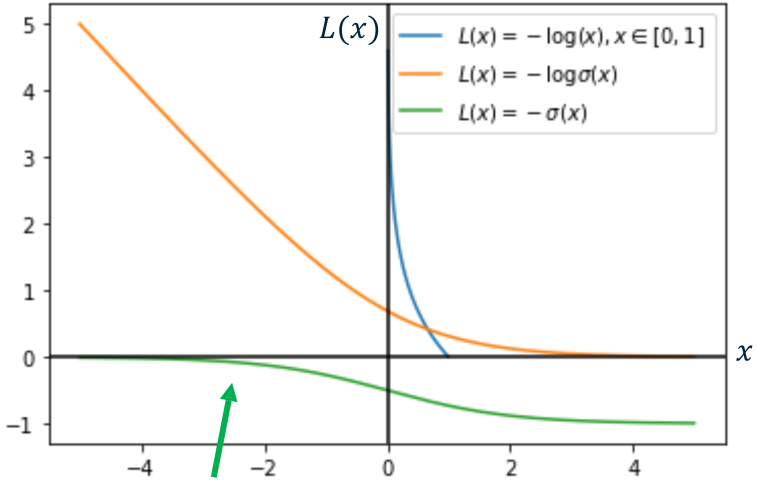
$$p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}$$

Use logistic function as example. Same as general softmax but for binary classification

$$\sigma(x) = \frac{e^x}{1 + e^x}$$

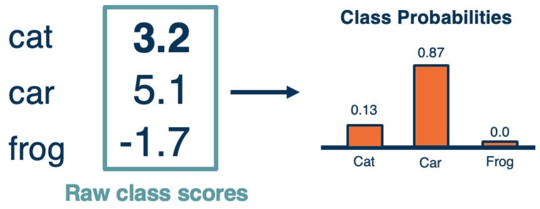
Consider the following three basis for NLL:

1. Squash and clip network value (x) to (0, 1]
2. (Negative) logistic function
3. NLL with logistic function




2. Negative likelihood w/ logistic function: saturated loss when classifier is very wrong

# Q: Why softmax?




Why this?

$$\boxed{p_\theta(Y = y_i | X = x_i) = \frac{e^{s_{y_i}}}{\sum_j e^{s_j}}}$$

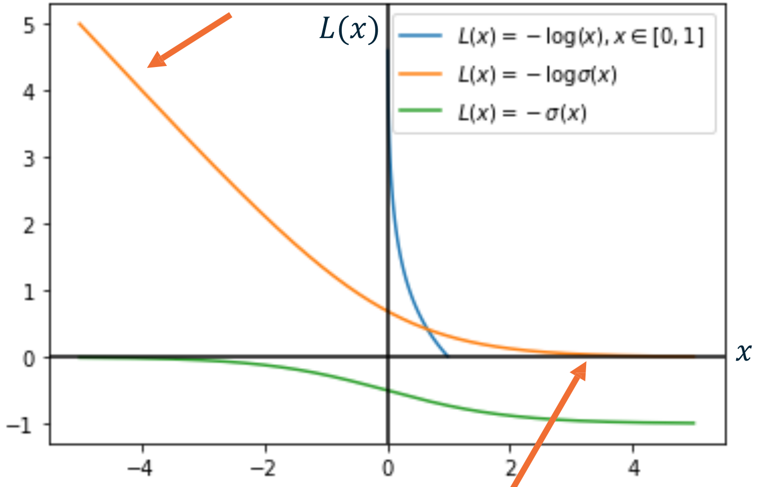
Use logistic function as example. Same as general softmax but for binary classification

$$\sigma(x) = \frac{e^x}{1 + e^x}$$

Consider the following three basis for NLL:

1. Squash and clip network value (x) to (0, 1]
2. (Negative) logistic function
3. NLL with logistic function

3. NLL w/ logistic: Strong guidance when classifier is wrong




Only saturate at convergence, e.g. $\sigma(3) \approx 0.95$

Softmax is a normalization function that behaves well with Cross Entropy Loss.

# So, what is a loss function?

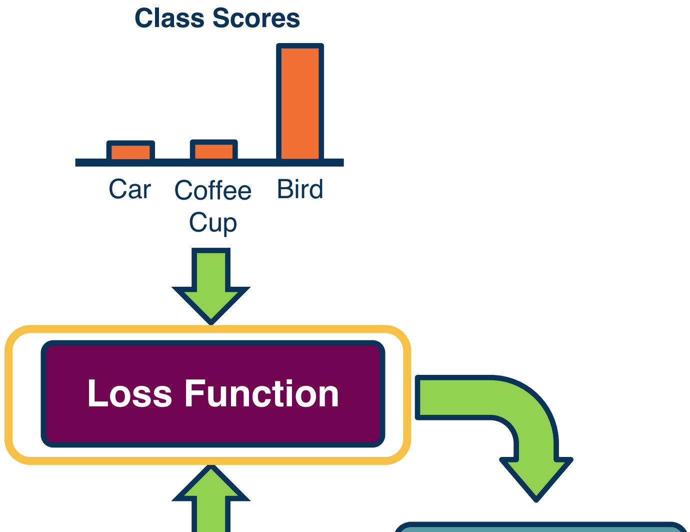
- In this context, it's a function that scores how well a **model** performs on a task. We often focus on the parameters rather than the hypothesis class.
- If $L(\theta_1, data) < L(\theta_2, data)$, then $\theta_1$ is considered better.

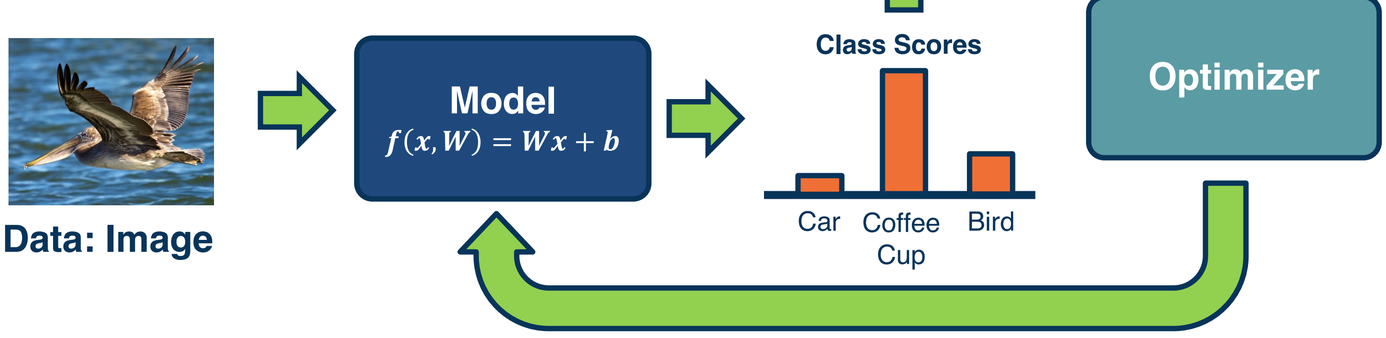
- **Losses** are different than **metrics.** Loss functions are designed for optimization, which require properties like differentiability and smoothness.
- Example: CrossEntropy is a loss function for the multi-class classification task. Classification accuracy (how often the model is correct) is the metric.
- Losses can be used as metrics but are often not very interpretable.
- Losses can always be used as metrics, but metrics often cannot be used as loss functions (e.g., classification accuracy is not differentiable).

# Summary: SVM and Softmax Classifier

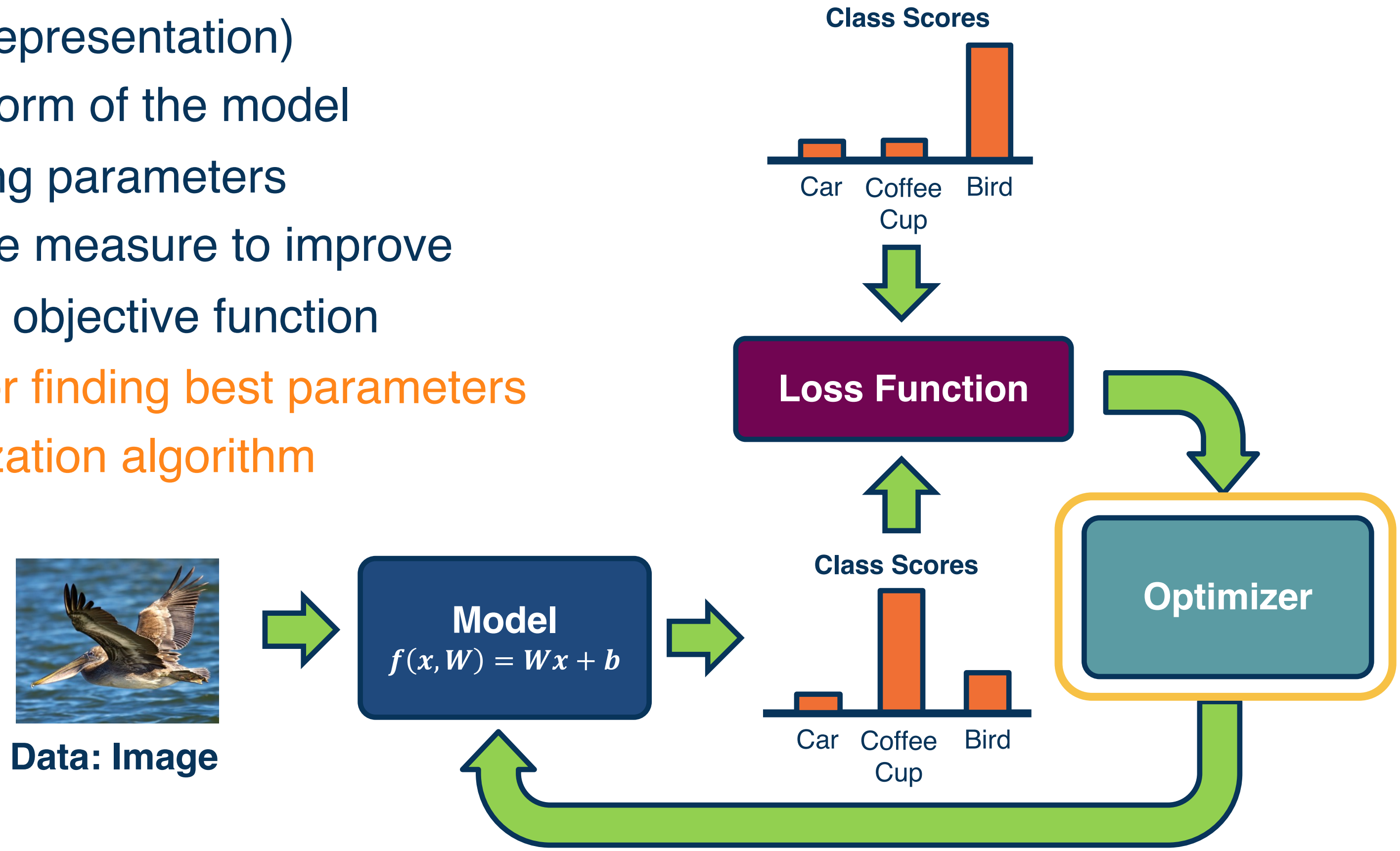
- Loss function: performance measure to improve
  - Find weights that better satisfies the objective
- Multiclass SVM Classifier
  - Predicts class score
  - Hinge loss: "maximum margin" objective: $L_i = \sum_{j \neq y_i} max(0, s_j - s_{y_i} + 1)$
- Regularization
  - Prevent overly complex function that only works well on the training set
- Softmax Classifier
  - Predicts class probabilities
  - To train softmax classifiers: use NLL and Cross Entropy Loss

- Input (and representation)
- Functional form of the model
    - Including parameters
- Performance measure to improve
    - Loss or objective function
- Algorithm for finding best parameters
    - Optimization algorithm



- Input (and representation)
- Functional form of the model
  - Including parameters
- Performance measure to improve
  - Loss or objective function
- Algorithm for finding best parameters
  - Optimization algorithm



# Strategy #1: A first very bad idea solution: **Random search**

```
# assume X_train is the data where each column is an example (e.g. 3073 x 50,000)
# assume Y_train are the labels (e.g. 1D array of 50,000)
# assume the function L evaluates the loss function

bestloss = float("inf") # Python assigns the highest possible float value
for num in xrange(1000):
  W = np.random.randn(10, 3073) * 0.0001 # generate random parameters
  loss = L(X_train, Y_train, W) # get the loss over the entire training set
  if loss < bestloss: # keep track of the best solution
    bestloss = loss
    bestW = W
  print 'in attempt %d the loss was %f, best %f' % (num, loss, bestloss)

# prints:
# in attempt 0 the loss was 9.401632, best 9.401632
# in attempt 1 the loss was 8.959668, best 8.959668
# in attempt 2 the loss was 9.044034, best 8.959668
# in attempt 3 the loss was 9.278948, best 8.959668
# in attempt 4 the loss was 8.857370, best 8.857370
# in attempt 5 the loss was 8.943151, best 8.857370
# in attempt 6 the loss was 8.605604, best 8.605604
# ... (trunctated: continues for 1000 lines)
```

# Lets see how well this works on the test set...

```
# Assume X_test is [3073 x 10000], Y_test [10000 x 1]
scores = Wbest.dot(Xte_cols) # 10 x 10000, the class scores for all test examples
# find the index with max score in each column (the predicted class)
Yte_predict = np.argmax(scores, axis = 0)
# and calculate accuracy (fraction of predictions that are correct)
np.mean(Yte_predict == Yte)
# returns 0.1555
```

15.5% accuracy! not bad!
(SOTA is ~99.7%)

*Adapted from from CS 231n slides*

Given a model and loss function, finding the best set of weights is a **search problem**

- Find the best combination of weights that minimizes our loss function

**Several classes of methods:**

- Random search
- Genetic algorithms (population-based search)
- Gradient-based optimization

In deep learning, **gradient-based methods are dominant** although not the only approach possible

$$\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1m} & b1 \\ w_{21} & w_{22} & \cdots & w_{2m} & b2 \\ w_{21} & w_{22} & \cdots & w_{3m} & b3 \end{bmatrix}$$

**Gradient**

**Loss**

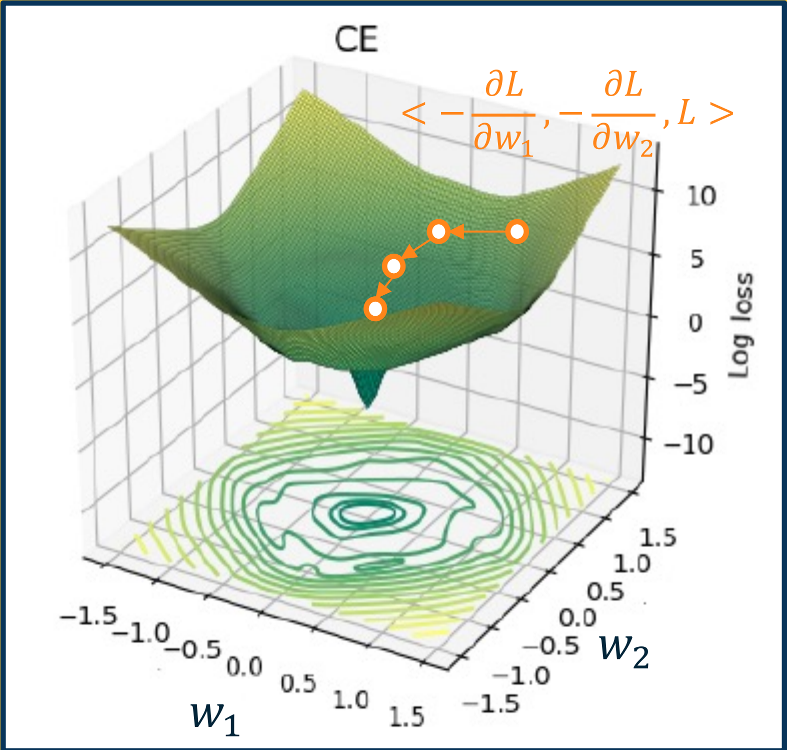
1. Calculate the gradients of a loss function with respect to a set of parameters (w's).
2. Update the parameters towards the gradient direction that minimizes the loss.

# Gradient Descent: Follow the Slope!

## As weights change, the gradients change as well

- This is often somewhat-smooth locally, so small changes in weights produce small changes in the loss

We can therefore think about **iterative algorithms** that take **current values of weights and modify them** a bit



- We can find the steepest descent direction by computing the **derivative:**

$$\frac{\partial f}{\partial w} = \lim_{h \to 0} \frac{f(w+h) - f(w)}{h}$$

- **Gradient** is multi-dimensional derivatives
- Notation: $\frac{\partial f}{\partial w}$ is the gradient of $f$ (e.g., a loss function) with respect to variable $w$ (e.g., a weight vector).
- $\frac{\partial f}{\partial w}$ is of the **same shape** as $w$
- **Intuitively:** Measures how the *output* changes as the variable $w$ changes by a small step size
- Steepest descent direction is the **negative gradient**
- **Gradient descent:** Minimize loss by changing parameters

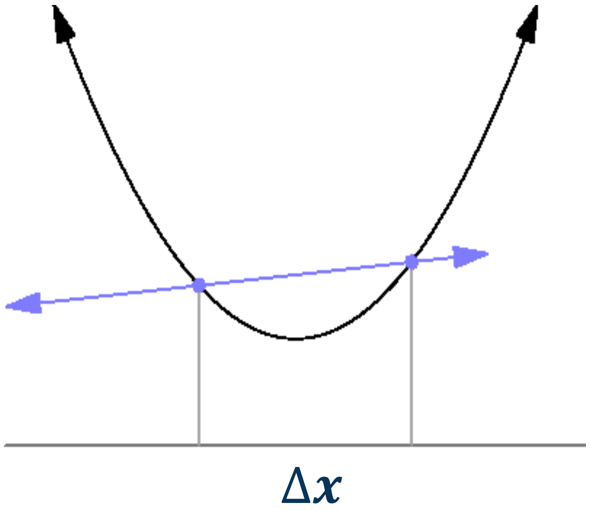



*Image and equation from: https://en.wikipedia.org/wiki/Derivative#/media/File:Tangent_animation.gif*

# Derivatives

# Calculate gradients: finite differences

**current W:**

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
**loss 1.25347**

**gradient dW:**

[?,
?,
?,
?,
?,
?,
?,
?,
?,…]

# Calculate gradients: finite differences

| **current W:** | **W + h** (first dim)**:** | **gradient dW:** |
| --- | --- | --- |
| [0.34, | [0.34 + **0.0001**, | [?, |
| -1.11, | -1.11, | ?, |
| 0.78, | 0.78, | ?, |
| 0.12, | 0.12, | ?, |
| 0.55, | 0.55, | ?, |
| 2.81, | 2.81, | ?, |
| -3.1, | -3.1, | ?, |
| -1.5, | -1.5, | ?, |
| 0.33,…] | 0.33,…] | ?,…] |
| **loss 1.25347** | **loss 1.25322** | |

# Calculate gradients: finite differences

**current W:**

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
**loss 1.25347**

**W + h** (first dim):

[0.34 + **0.0001**,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
**loss 1.25322**

**gradient dW:**

[**-2.5**,
?,
?,

(1.25322 - 1.25347)/0.0001
= -2.5

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

?,
?,…]

# Calculate gradients: finite differences

| current W: | W + h (second dim): | gradient dW: |
|---|---|---|
| [0.34, | [0.34, | [-2.5, |
| -1.11, | -1.11 + **0.0001**, | ?, |
| 0.78, | 0.78, | ?, |
| 0.12, | 0.12, | ?, |
| 0.55, | 0.55, | ?, |
| 2.81, | 2.81, | ?, |
| -3.1, | -3.1, | ?, |
| -1.5, | -1.5, | ?, |
| 0.33,…] | 0.33,…] | ?,…] |
| **loss 1.25347** | **loss 1.25353** | |

# Calculate gradients: finite differences

**current W:**

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
**loss 1.25347**

**W + h** (second dim):

[0.34,
-1.11 + **0.0001**,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,…]
**loss 1.25353**

**gradient dW:**

[-2.5,
**0.6**,
?,
?,
?,…]

(1.25353 - 1.25347)/0.0001
= 0.6

$$\boxed{\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

# Calculate gradients: finite differences

| current W: | W + h (third dim): | gradient dW: |
|---|---|---|
| [0.34, | [0.34, | [-2.5, |
| -1.11, | -1.11, | 0.6, |
| 0.78, | 0.78 + **0.0001**, | ?, |
| 0.12, | 0.12, | ?, |
| 0.55, | 0.55, | ?, |
| 2.81, | 2.81, | ?, |
| -3.1, | -3.1, | ?, |
| -1.5, | -1.5, | ?, |
| 0.33,…] | 0.33,…] | ?,…] |
| **loss 1.25347** | **loss 1.25347** | |

# Calculate gradients: finite differences

| **current W:** | **W + h** (third dim)**:** | **gradient dW:** |
|---|---|---|
| [0.34, | [0.34, | [-2.5, |
| -1.11, | -1.11, | 0.6, |
| 0.78, | 0.78 + **0.0001**, | **0**, |
| 0.12, | 0.12, | ?, |
| 0.55, | 0.55, | |
| 2.81, | 2.81, | |
| -3.1, | -3.1, | |
| -1.5, | -1.5, | |
| 0.33,…] | 0.33,…] | ?,…] |
| **loss 1.25347** | **loss 1.25347** | |

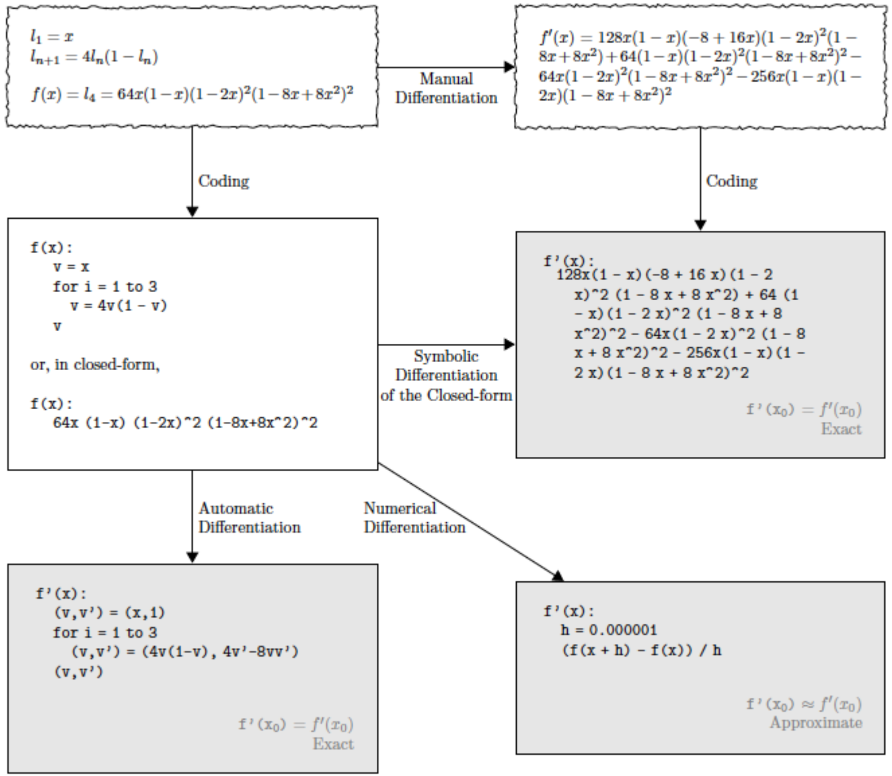
(1.25347 - 1.25347)/0.0001
= 0

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

# Several ways to compute $\frac{\partial L}{\partial w_i}$

- Manual differentiation
- Symbolic differentiation
- Numerical differentiation
- **Automatic differentiation**

More on **autodiff**:
https://www.cs.toronto.edu/~rgrosse/courses/csc421_2019/readings/L06%20Automatic%20Differentiation.pdf



# Numerical vs Analytic Gradients

$$\frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

**Numerical gradient**: slow, approximate, easy to implement
**Analytic gradient**: fast, exact, error-prone (if implemented from scratch)

Almost all differentiable functions that you can think of have analytical gradients implemented in popular libraries, e.g., PyTorch, TensorFlow.

If you want to derive your own gradients, check your implementation with numerical gradient.
This is called a **gradient check.**

## The gradient descent algorithm

- 1. Choose a model: $\boldsymbol{f(x, W)} = \mathrm{Wx}$
- 2. Choose loss function: $\boldsymbol{L_i = |y - Wx_i|^2}$
- 3. Calculate partial derivative for each parameter: $\boldsymbol{\frac{\partial L}{\partial w_i}}$
- 4. Update the parameters: $\boldsymbol{w_i = w_i - \frac{\partial L}{\partial w_i}}$
- 5. Add learning rate to prevent too big of a step: $\boldsymbol{w_i = w_i - \alpha \frac{\partial L}{\partial w_i}}$
- **Repeat 3-5**

# Composing simple functions creates complex analytical gradients

$\sin(x)$ $\log(x)$ $\cos(x)$ $x^3$ $\exp(x)$

Compose into a complex function

$$-\log\left(\frac{1}{1+e^{-w\cdot x}}\right)$$

$$\boxed{w\cdot x} \xrightarrow{u} \boxed{\frac{1}{1+e^{-u}}} \xrightarrow{p} \boxed{-\log(p)} \xrightarrow{L}$$

*Adapted from slides by: Marc'Aurelio Ranzato, Yann LeCun*


$$\frac{\partial L}{\partial w} = \frac{\partial L}{\partial p}\frac{\partial p}{\partial u}\frac{\partial u}{\partial w}$$

Next time: Chain rule and Backpropagation!

*Adapted from slides by: Marc'Aurelio Ranzato, Yann LeCun*