Topics:

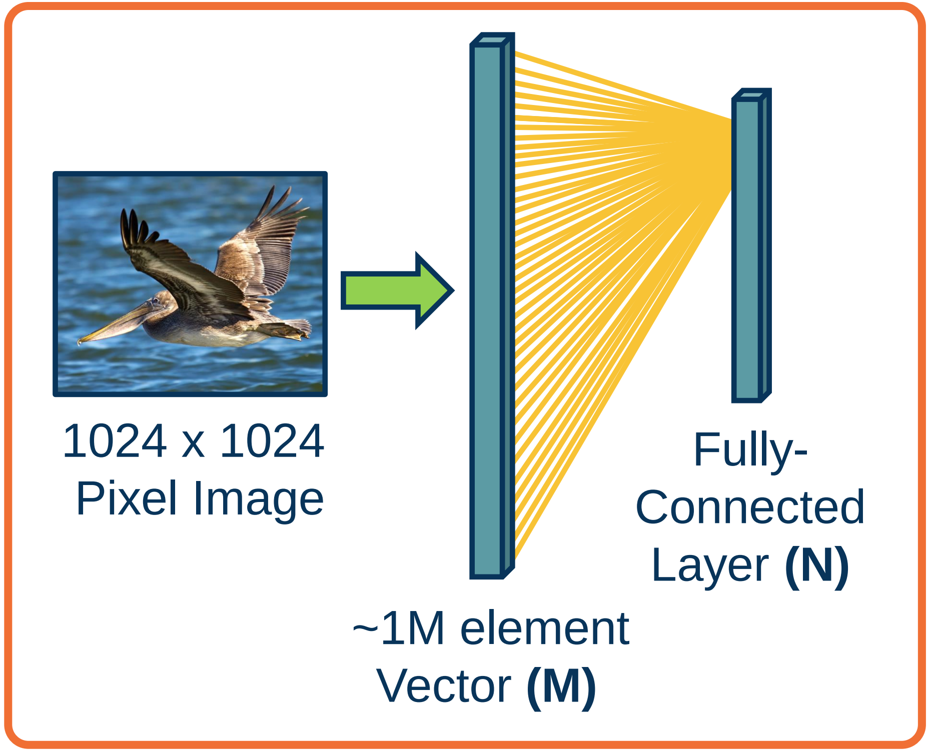
- Convolutional Neural Networks

CS 4644-DL / 7643-A

ZSOLT KIRA

- **Assignment 2**
  - Implement convolutional neural networks
  - Resources (in addition to lectures):
    - [DL book: Convolutional Networks]
    - CNN notes https://www.cc.gatech.edu/classes/AY2022/cs7643_spring/assets/L10_cnns_notes.pdf
    - Backprop notes https://www.cc.gatech.edu/classes/AY2022/cs7643_spring/assets/L10_cnns_backprop_notes.pdf
    - HW2 Tutorial (piazza @113)
    - Slower OMSCS lectures on dropbox: Module 2 Lessons 5-6 (M2L5/M2L6) (https://www.dropbox.com/sh/iviro188gq0b4vs/AADdHxX_Uy1TkpF_yvIzX0nPa?dl=0)
- **GPU resources**
  - **For assignments, can use CPU or Google Colab**
  - **Projects:**
    - **Google Cloud Credits**
    - **Working on Amazon AWS**

The connectivity in linear layers **doesn't always make sense**



How many parameters?

- M*N (weights) + N (bias)

Hundreds of millions of parameters **for just one layer**

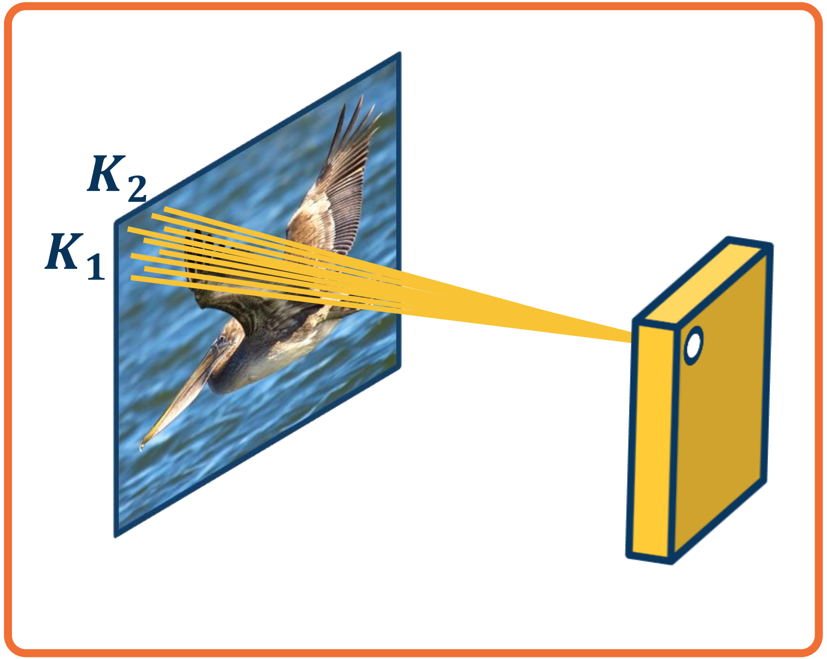
More parameters => More data needed

Is this necessary?

Image features are spatially localized!

- Smaller features repeated across the image
  - Edges
  - Color
  - Motifs (corners, etc.)
- No reason to believe one feature tends to appear in one location vs. another (stationarity)

Can we induce a *bias* in the design of a neural network layer to reflect this?



Each node only receives input from $K_1 \times K_2$ window (image patch)

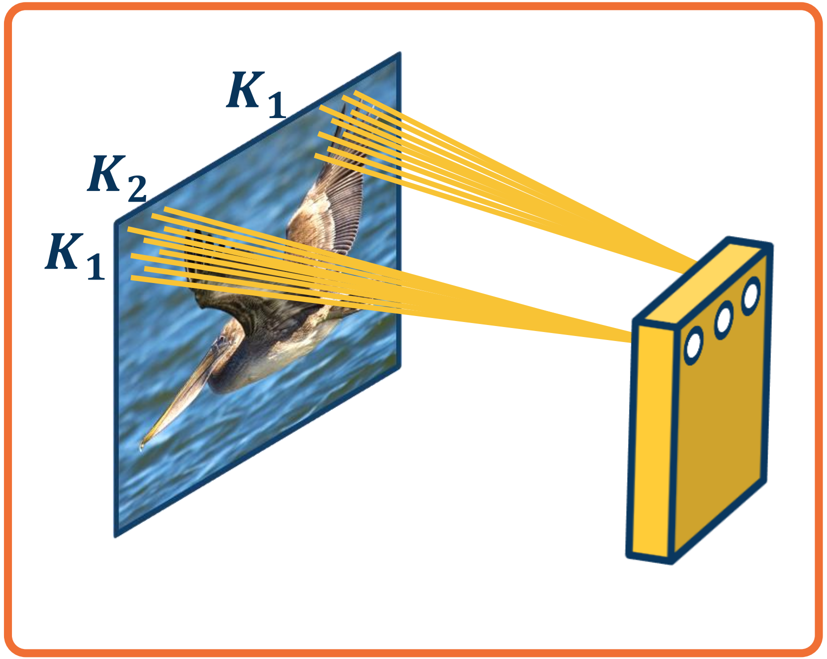
- Region from which a node receives input from is called its **receptive field**

Advantages:

- Reduce parameters to $(K_1 \times K_2 + 1) * N$ where N is number of output nodes
- Explicitly maintain spatial information

Do we need to learn location-specific features?

Idea 1: Receptive Fields



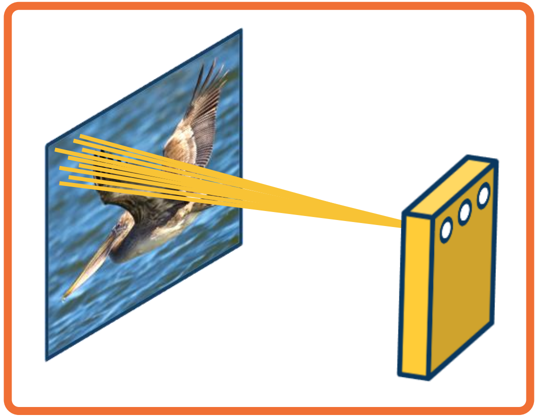
Nodes in different locations can **share** features

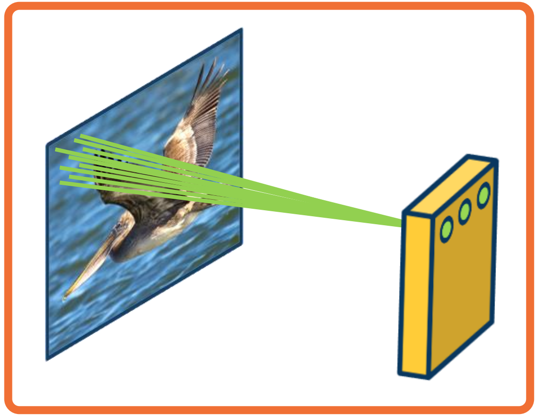
- No reason to think same feature (e.g. edge pattern) can't appear elsewhere
- Use same weights/parameters in computation graph **(shared weights)**

Advantages:

- Reduce parameters to $(K_1 \times K_2 + 1)$
- Explicitly maintain spatial information

Idea 2: Shared Weights



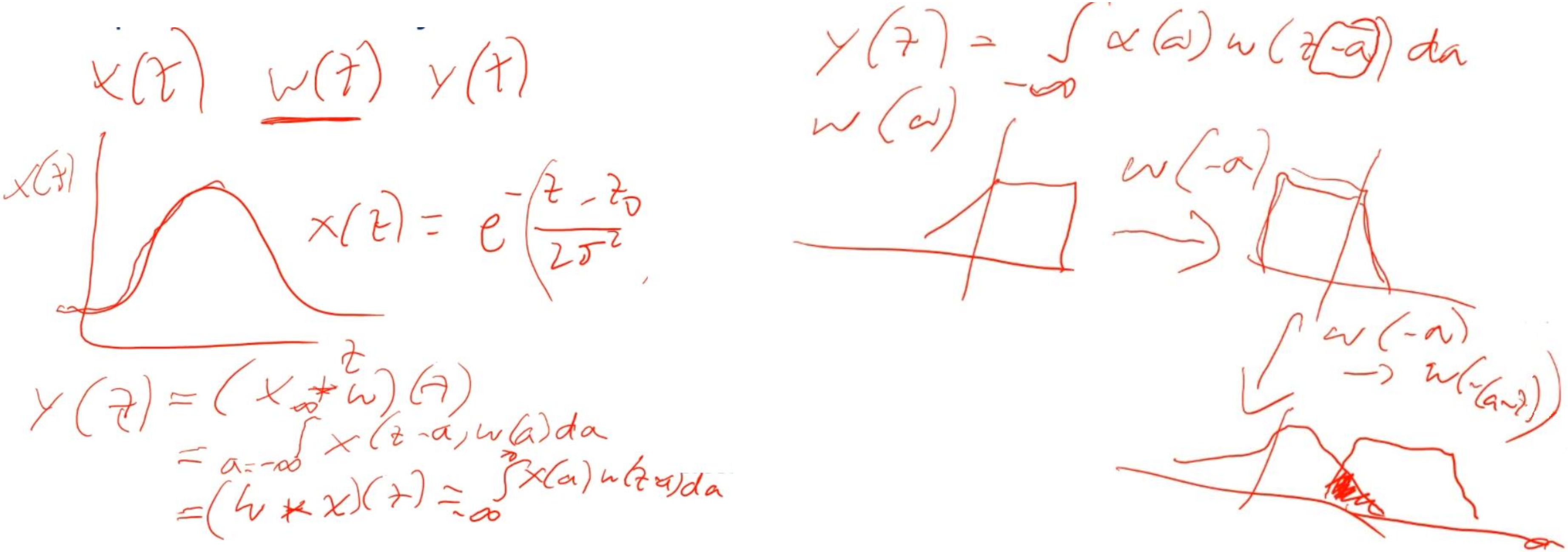


We can learn **many** such features for this one layer

- Weights are **not** shared across different feature extractors
- **Parameters:** $(\boldsymbol{K_1} \times \boldsymbol{K_2} + \mathbf{1}) * \boldsymbol{M}$ where $\boldsymbol{M}$ is number of features we want to learn

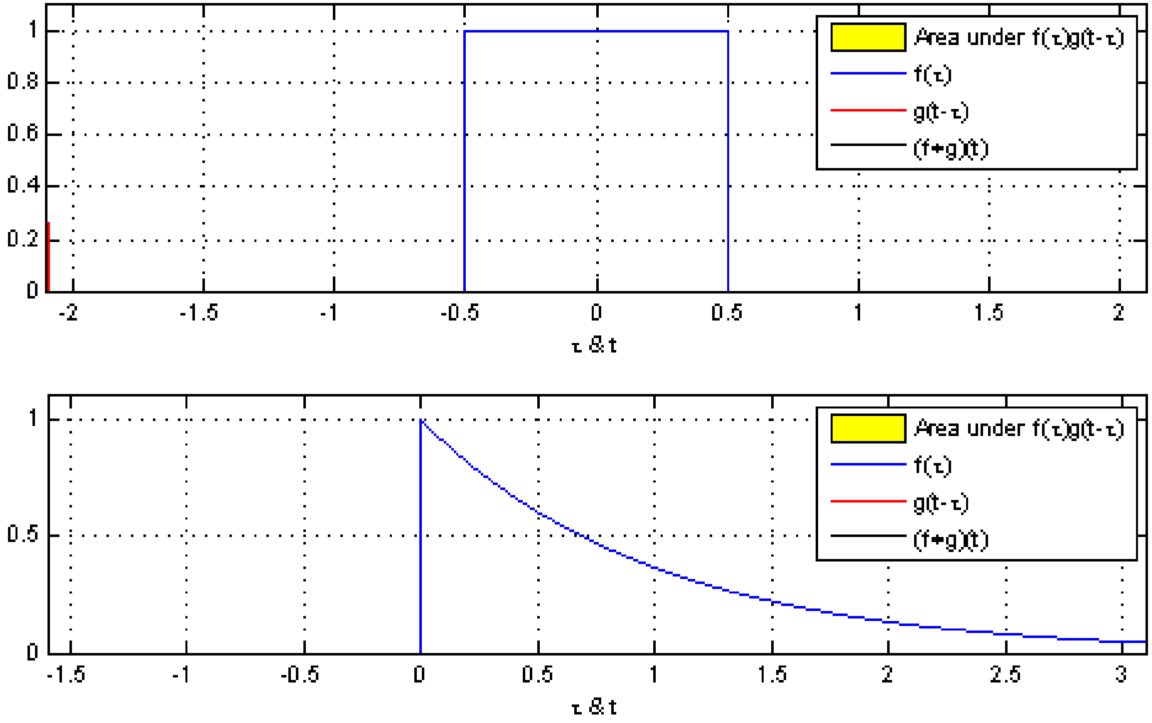
Idea 3: Learn Many Features

This operation is **extremely common** in electrical/computer engineering!



From https://en.wikipedia.org/wiki/Convolution

This operation is **extremely common** in electrical/computer engineering!



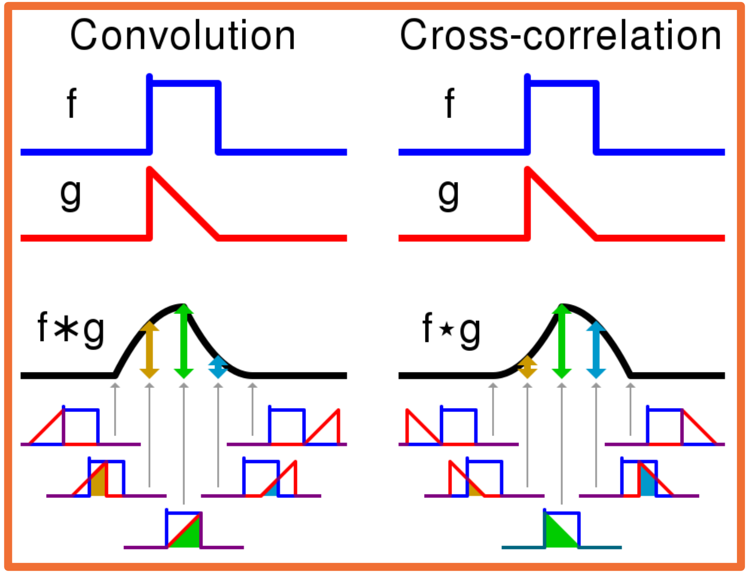
From https://en.wikipedia.org/wiki/Convolution

This operation is **extremely common** in electrical/computer engineering!

In mathematics and, in particular, functional analysis, **convolution** is a mathematical operation on two functions f and g producing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated.

Convolution is similar to **cross-correlation**.

It has **applications** that include probability, statistics, computer vision, image and signal processing, electrical engineering, and differential equations.



Visual comparison of **convolution** and **cross-correlation.**

From https://en.wikipedia.org/wiki/Convolution

Notation: $F \otimes (G \otimes I) = (F \otimes G) \otimes I$

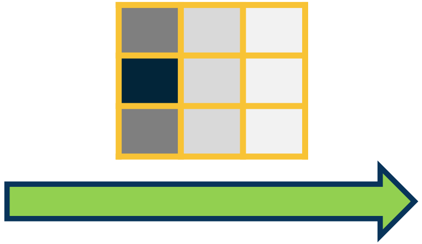
1D Convolution

$$y_k = \sum_{n=0}^{N-1} h_n \cdot x_{k-n}$$

$$y_0 = h_0 \cdot x_0$$
$$y_1 = h_1 \cdot x_0 + h_0 \cdot x_1$$
$$y_2 = h_2 \cdot x_0 + h_1 \cdot x_1 + h_0 \cdot x_2$$
$$y_3 = h_3 \cdot x_0 + h_2 \cdot x_1 + h_1 \cdot x_2 + h_0 \cdot x_3$$
$$\vdots$$

2D Convolution

$$K = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$



2D Convolution

Image

Kernel
(or filter)

$$K = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

Output /
filter /
feature map

2D Discrete Convolution

We will make this convolution operation **a layer** in the neural network

- Initialize kernel values randomly and optimize them!
- These are our parameters (plus a bias term per filter)

2D Convolution

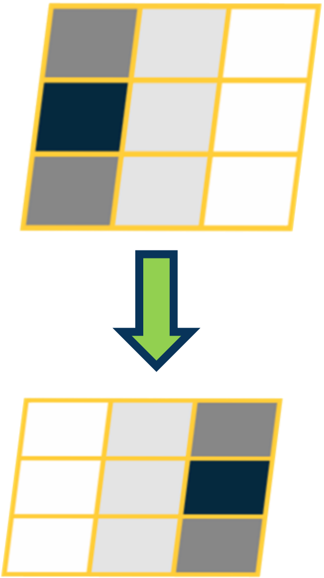
Image

Kernel (or filter)

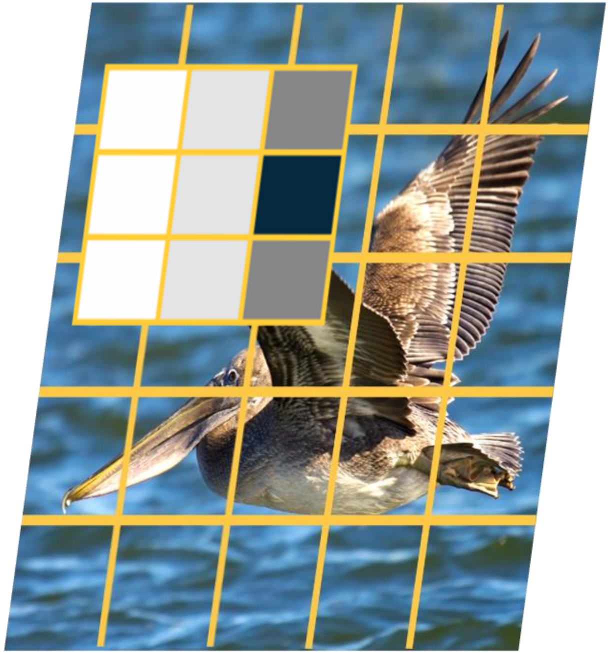
$$K = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

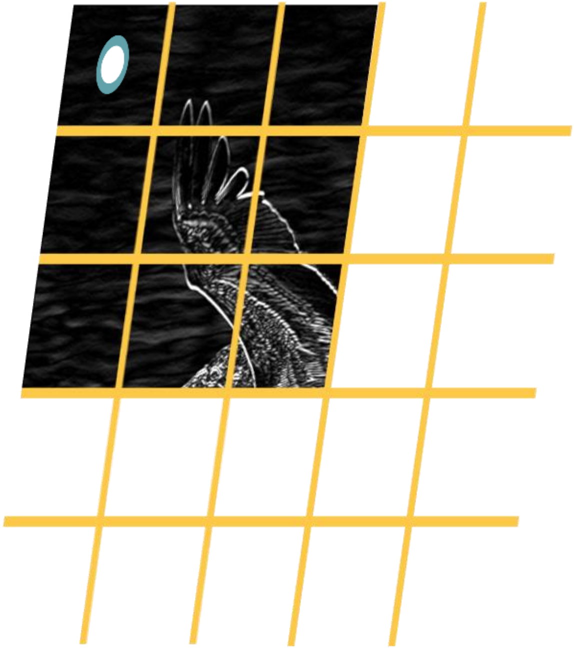
Output / filter / feature map

1. Flip kernel (rotate 180 degrees)

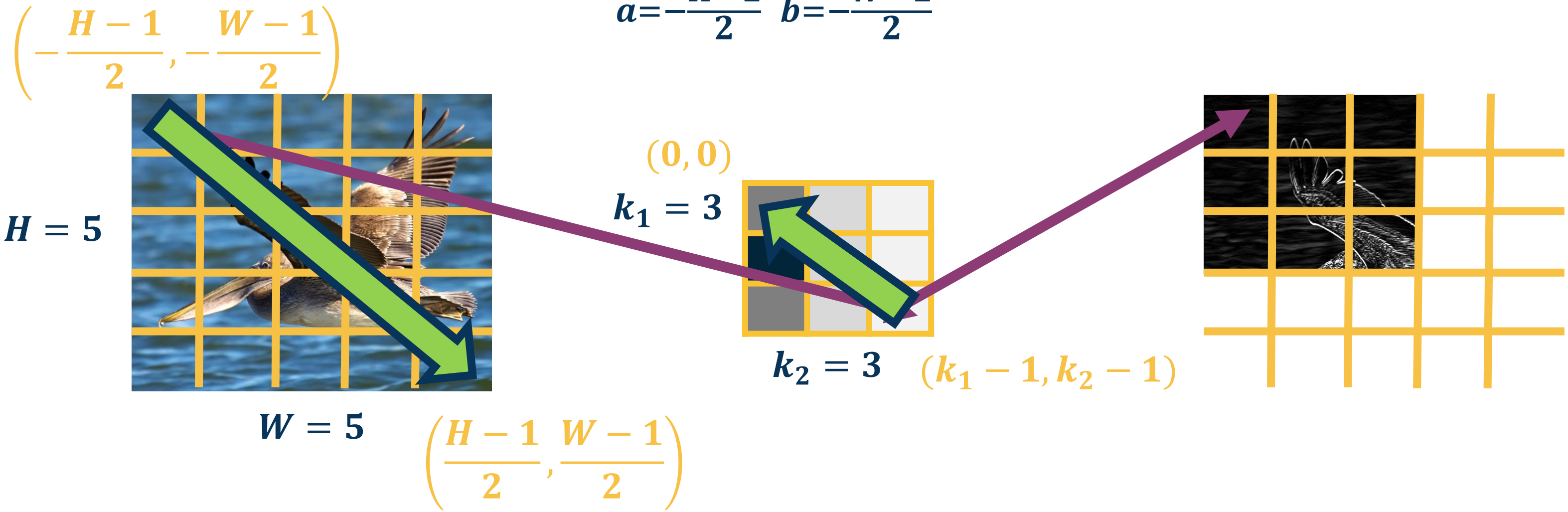


2. Stride along image





$$y(r,c) = (x * k)(r,c) = \sum_{a=-\frac{H-1}{2}}^{\frac{H-1}{2},} \sum_{b=-\frac{W-1}{2}}^{\frac{W-1}{2},} x(a,b)\, k(r-a, c-b)$$



$$y(0,0) = x(-2,-2)k(2,2) + x(-2,-1)k(2,1) + x(-2,0)k(2,0) + x(-2,1)k(2,-1) + x(-2,2)k(2,-2) + \ldots$$

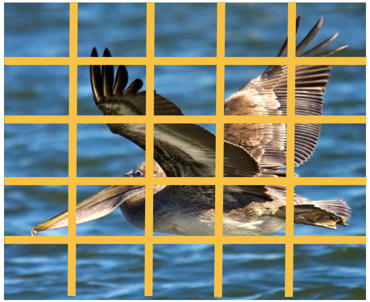
Mathematics of Discrete 2D Convolution

Centering Around the Kernel

$$y(r,c) = (x * k)(r,c) = \sum_{a=-\frac{K_1-1}{2}}^{\frac{k_1-1}{2},} \sum_{b=-\frac{k_2-1}{2}}^{\frac{k_2-1}{2},} x(r-a, c-b)\, k(a,b)$$

$(0, 0)$

$H = 5$

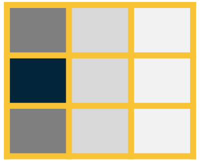


$W = 5$

$(H-1, W-1)$

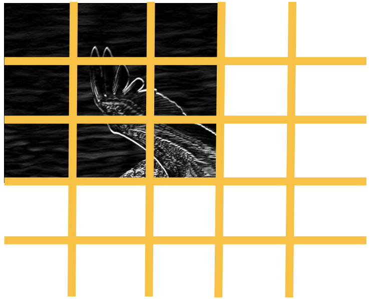
$\left(-\frac{k_1-1}{2}, -\frac{k_2-1}{2}\right)$

$k_1 = 3$



$k_2 = 3$

$\left(\frac{k_1-1}{2}, \frac{k_2-1}{2}\right)$



As we have seen:

- **Convolution:** Start at end of kernel and move back
- **Cross-correlation:** Start in the beginning of kernel and move forward (same as for image)

An **intuitive interpretation** of the relationship:

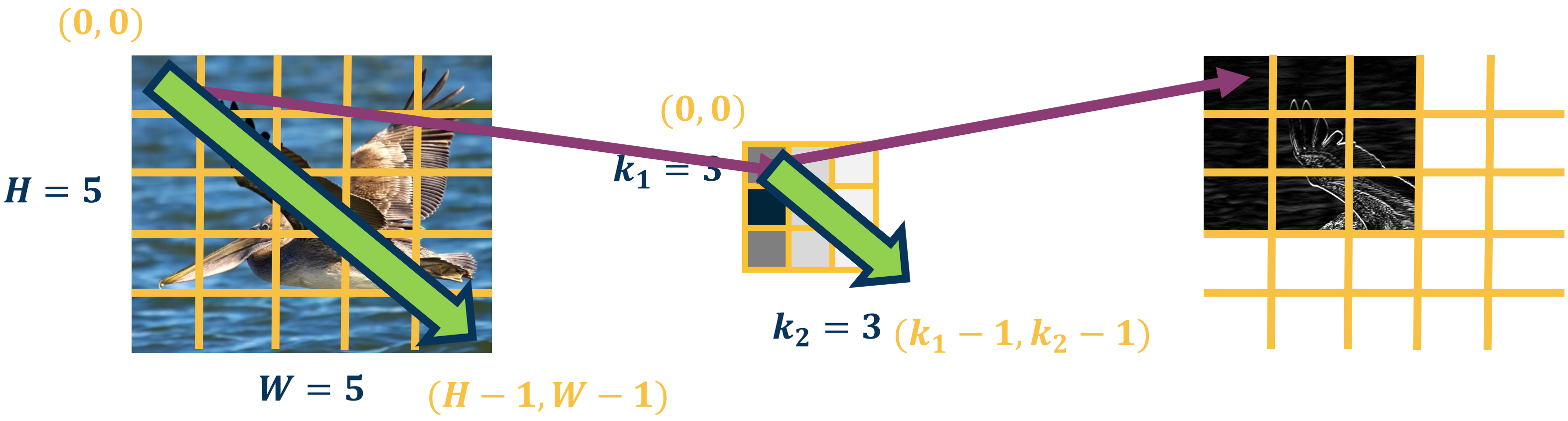
- Take the kernel, and rotate 180 degrees along center (sometimes referred to as “flip”)
- Perform cross-correlation
- (Just dot-product filter with image!)

$$K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\Downarrow$$

$$K' = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$y(r,c) = (x * k)(r,c) = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} x(r+a, c+b)\, k(a,b)$$

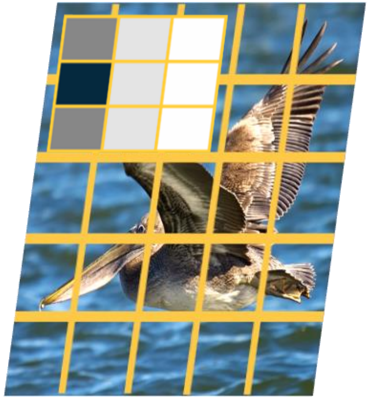


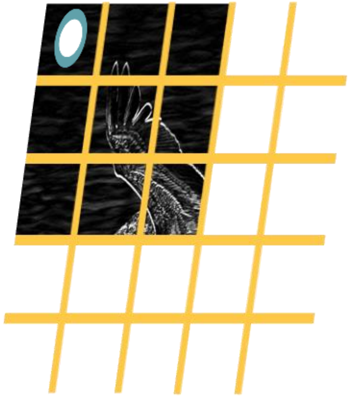
Since we will be learning these kernels, this change does not matter!

Cross-Correlation

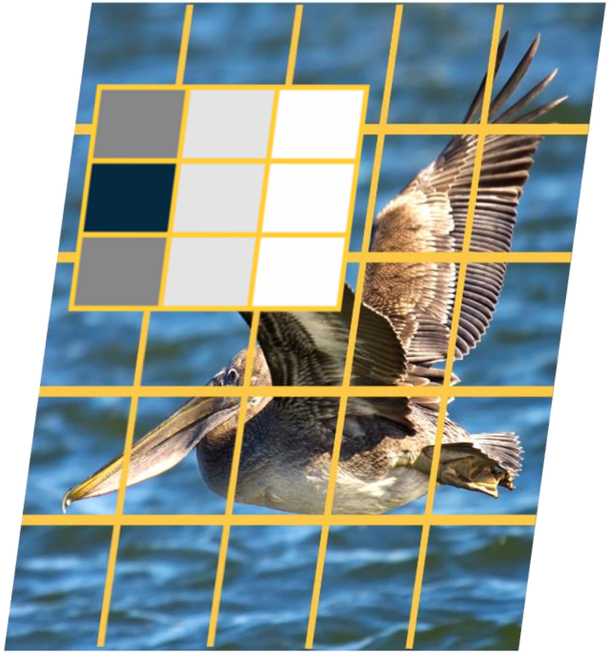
$$X(0:2,0:2) = \begin{bmatrix} 200 & 150 & 150 \\ 100 & 50 & 100 \\ 25 & 25 & 10 \end{bmatrix} \qquad K' = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow X(0:2,0:2) \cdot K' = 65 + \text{bias}$$

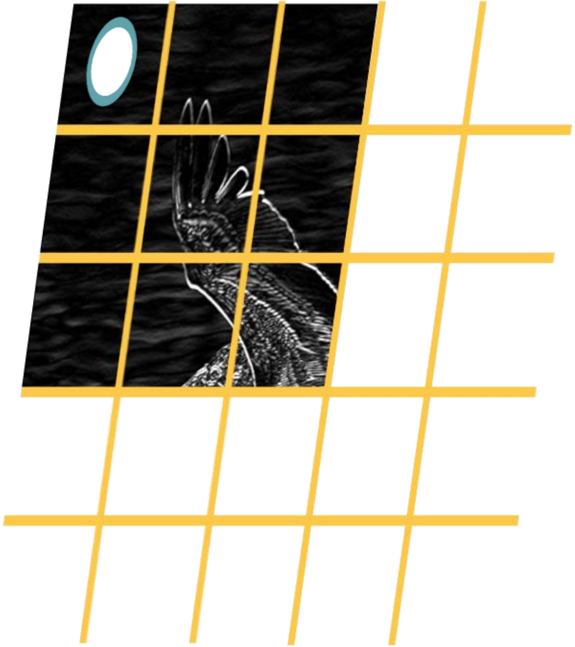
Dot product
(element-wise multiply and sum)



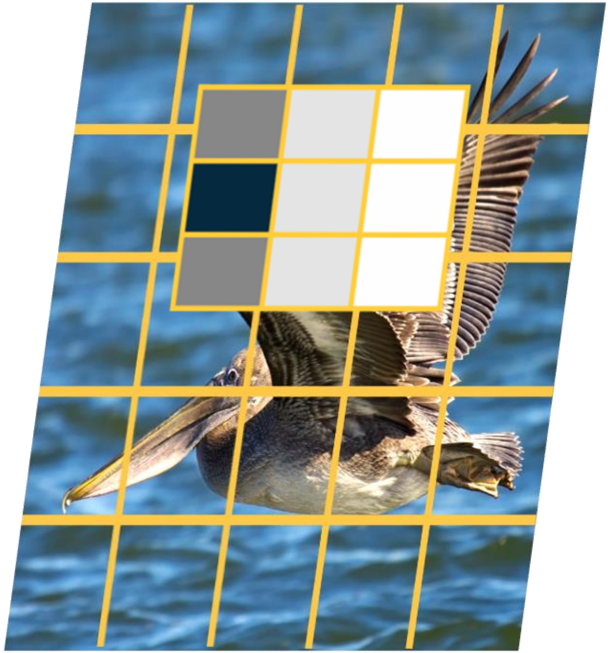


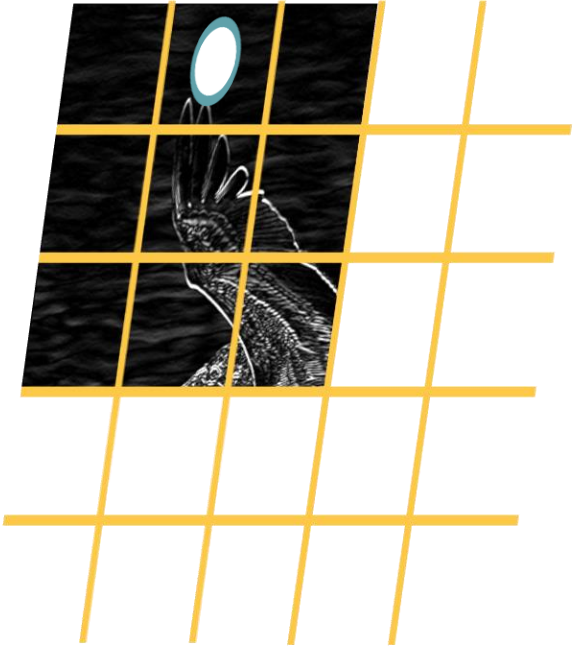
Cross-Correlation



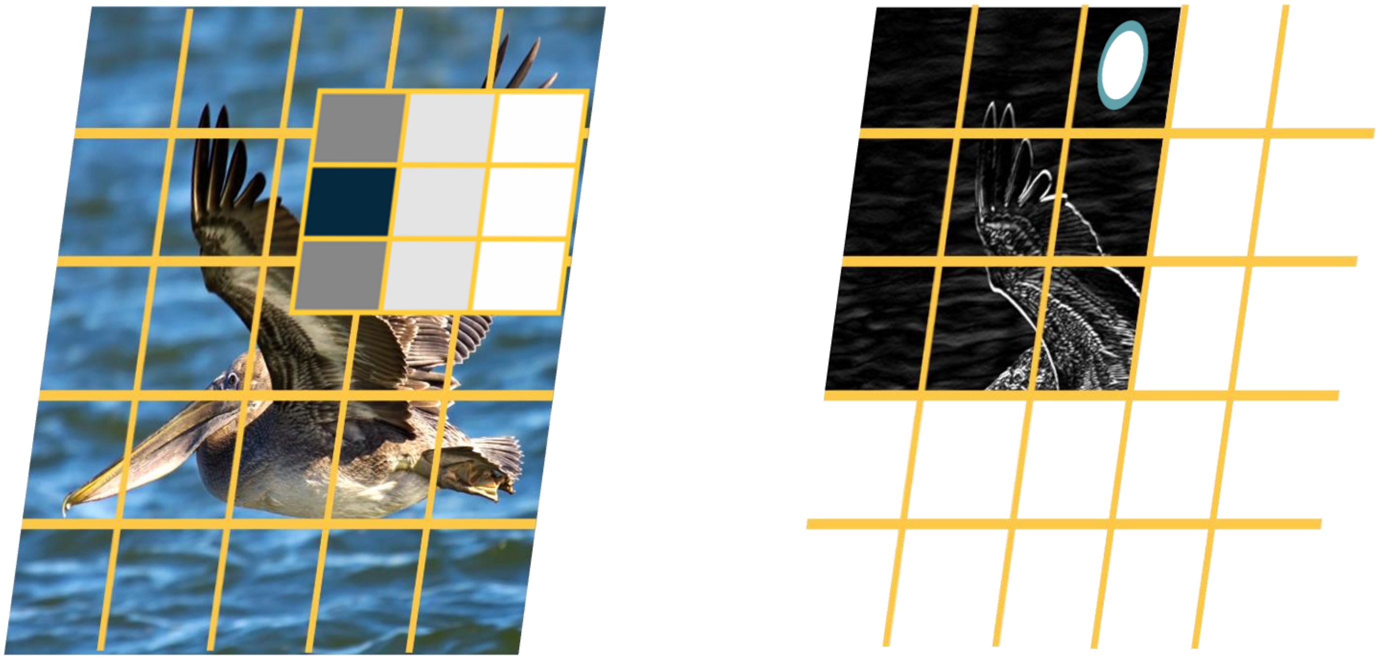


Convolution and Cross-Correlation

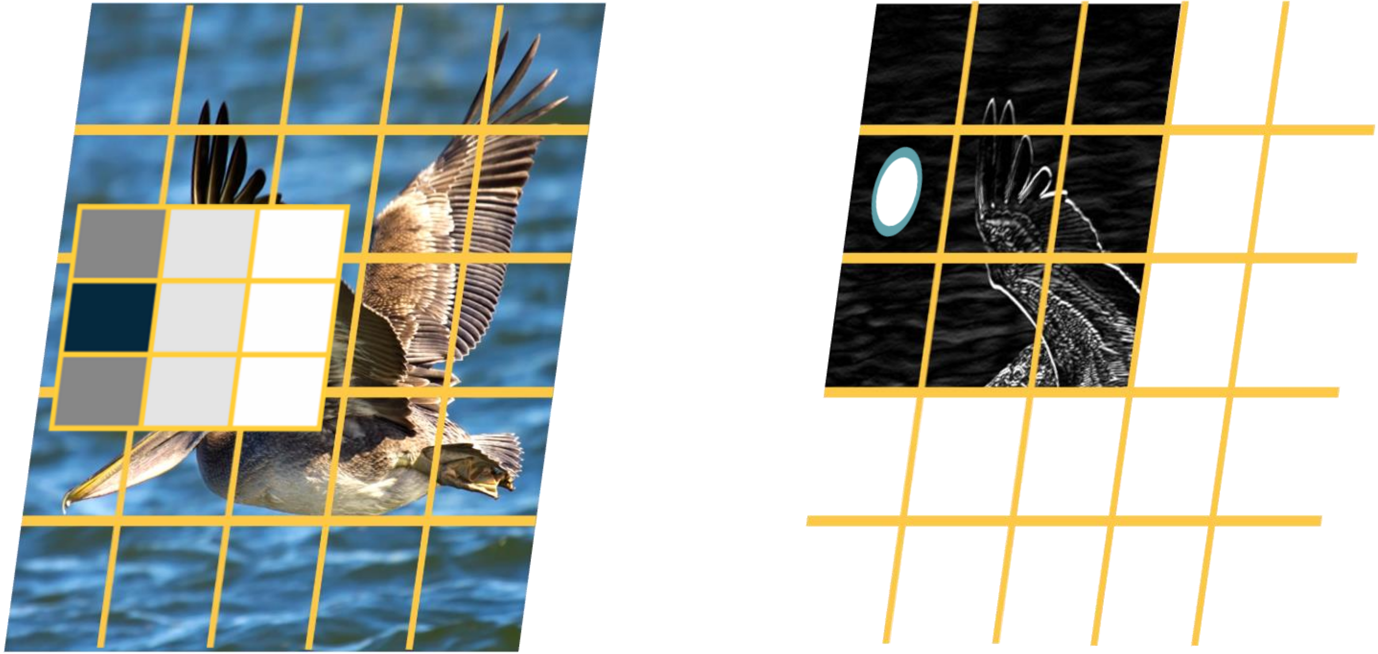




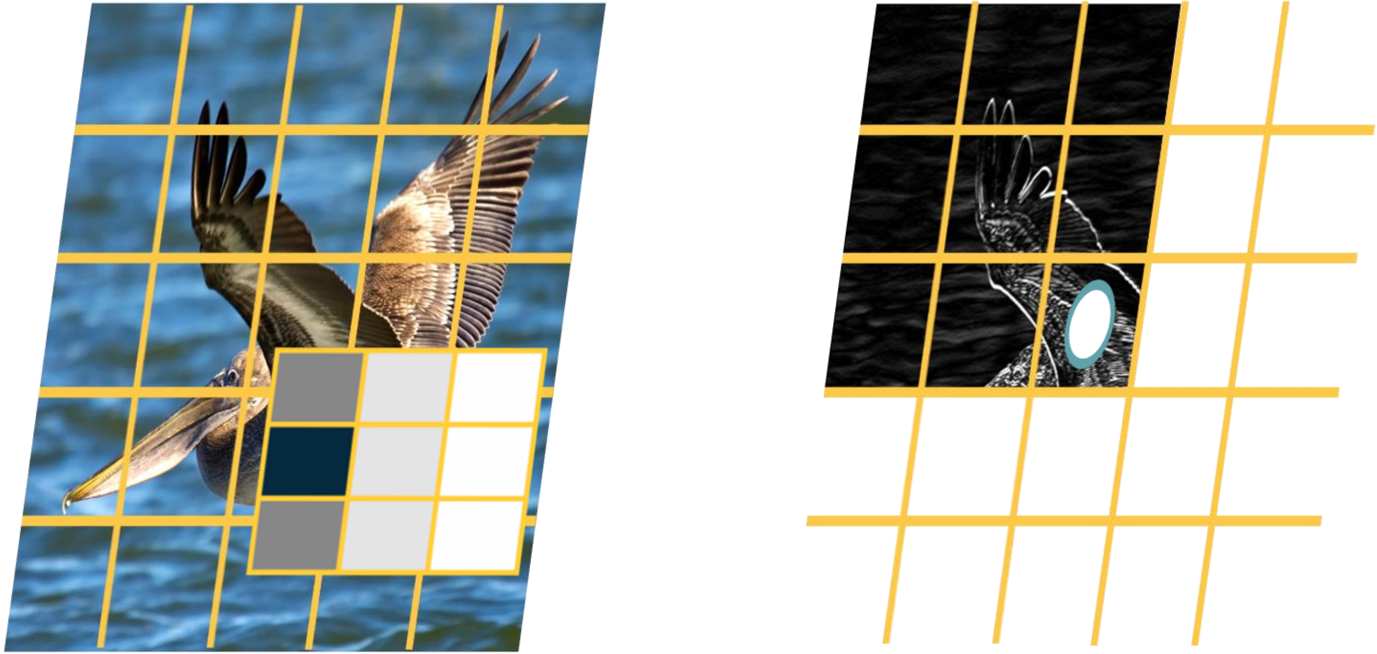
Convolution and Cross-Correlation



Convolution and Cross-Correlation



Convolution and Cross-Correlation



Convolution and Cross-Correlation

Why Bother with Convolutions?

Convolutions are just **simple linear operations**

Why bother with this and not just say it's a linear layer with small receptive field?

- There is a **duality** between them during backpropagation
- Convolutions have **various mathematical properties** people care about
- This is **historically** how it was inspired

Input & Output Sizes

Convolution Layer Hyper-Parameters

Parameters

- **in_channels** (*int*) – Number of channels in the input image
- **out_channels** (*int*) – Number of channels produced by the convolution
- **kernel_size** (*int or tuple*) – Size of the convolving kernel
- **stride** (*int or tuple, optional*) – Stride of the convolution. Default: 1
- **padding** (*int or tuple, optional*) – Zero-padding added to both sides of the input. Default: 0
- **padding_mode** (*string, optional*) – `'zeros'`, `'reflect'`, `'replicate'` or `'circular'`. Default: `'zeros'`

Convolution operations have several hyper-parameters

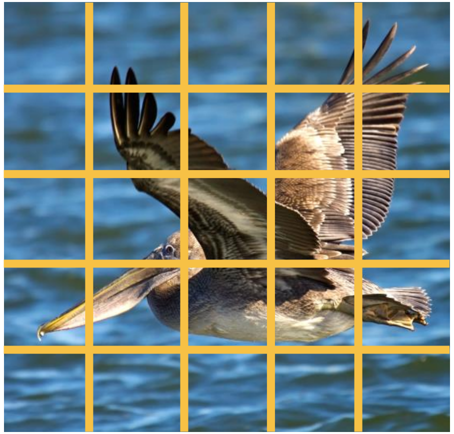
From: https://pytorch.org/docs/stable/generated/torch.nn.Conv2d.html#torch.nn.Conv2d

Output size of vanilla convolution operation is $(H - k_1 + 1) \times (W - k_2 + 1)$

- This is called a **“valid” convolution** and only applies kernel within image

$(0, 0)$

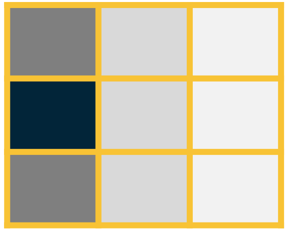
$H = 5$



$W = 5$ $(H-1, W-1)$

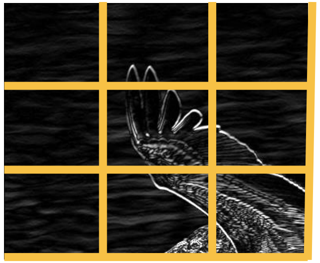
$(0, 0)$

$k_1 = 3$



$k_2 = 3$ $(k_1 - 1, k_2 - 1)$

$H - k_1 + 1$

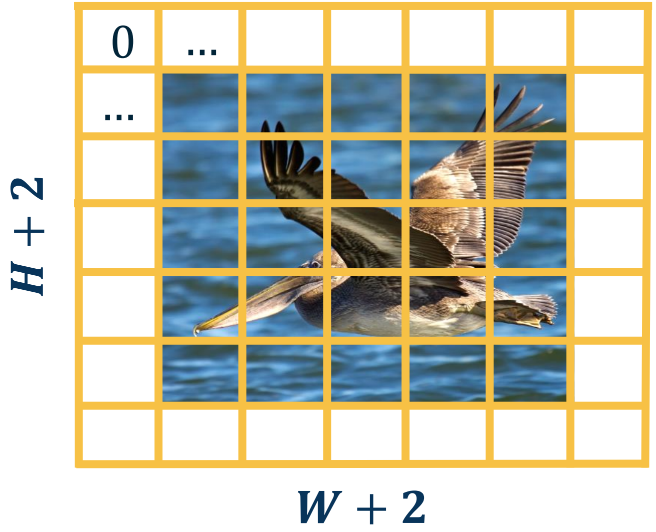


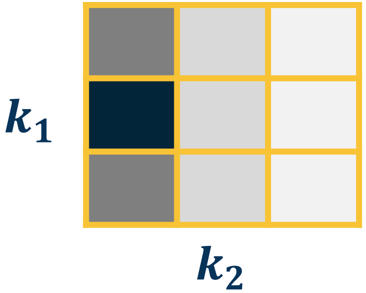
$W - k_2 + 1$

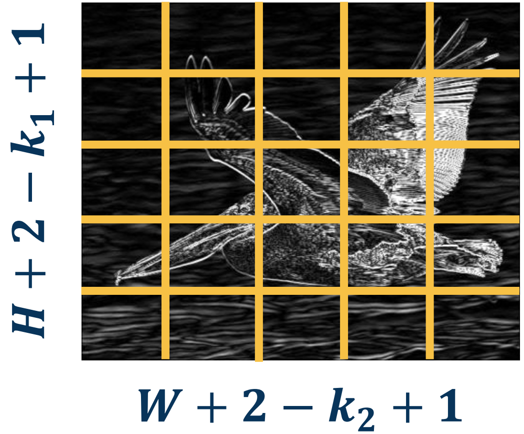
Valid Convolution

We can **pad the images** to make the output the same size:

- Zeros, mirrored image, etc.
- Note padding often refers to pixels added to **one size $(\mathbf{P = 1\ here})$**





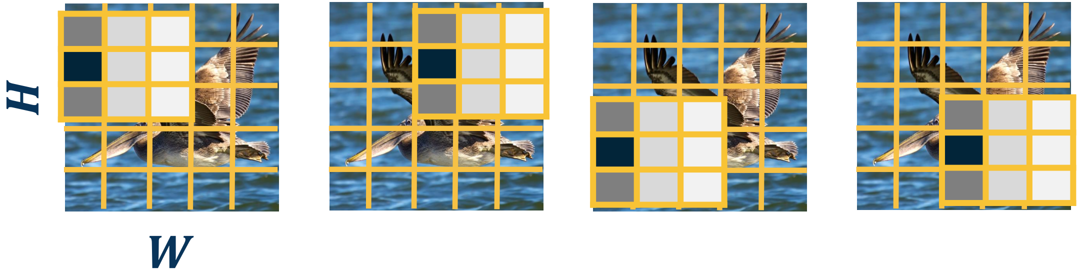


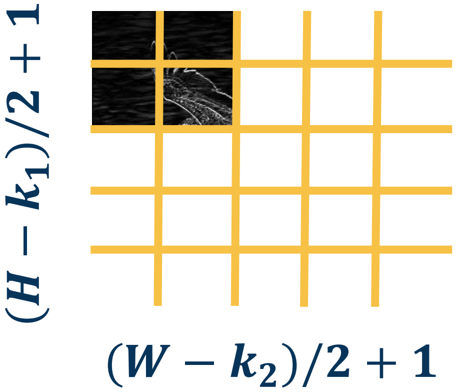
Adding Padding

We can move the filter along the image using larger steps **(stride)**

- This can potentially result in **loss of information**
- Can be used for **dimensionality reduction** (not recommended)

Stride = 2 (every other pixel)





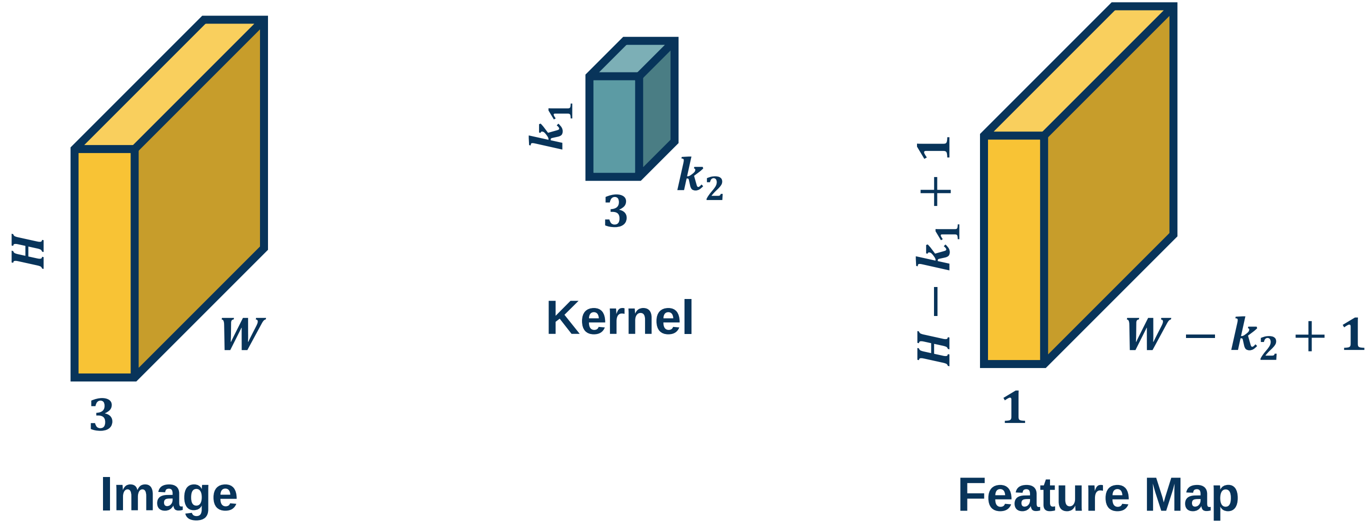
Stride

Stride can result in **skipped pixels**, e.g. stride of 3 for 5x5 input

Invalid Stride

We have shown inputs as a **one-channel image** but in reality they have three channels (red, green, blue)

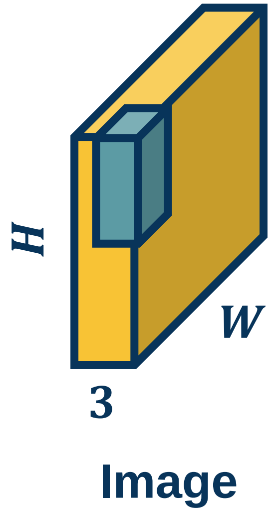
- In such cases, we have **3-channel kernels**!



Multi-Channel Inputs

We have shown inputs as a **one-channel image** but in reality they have three channels (red, green, blue)

- In such cases, we have **3-channel kernels**!



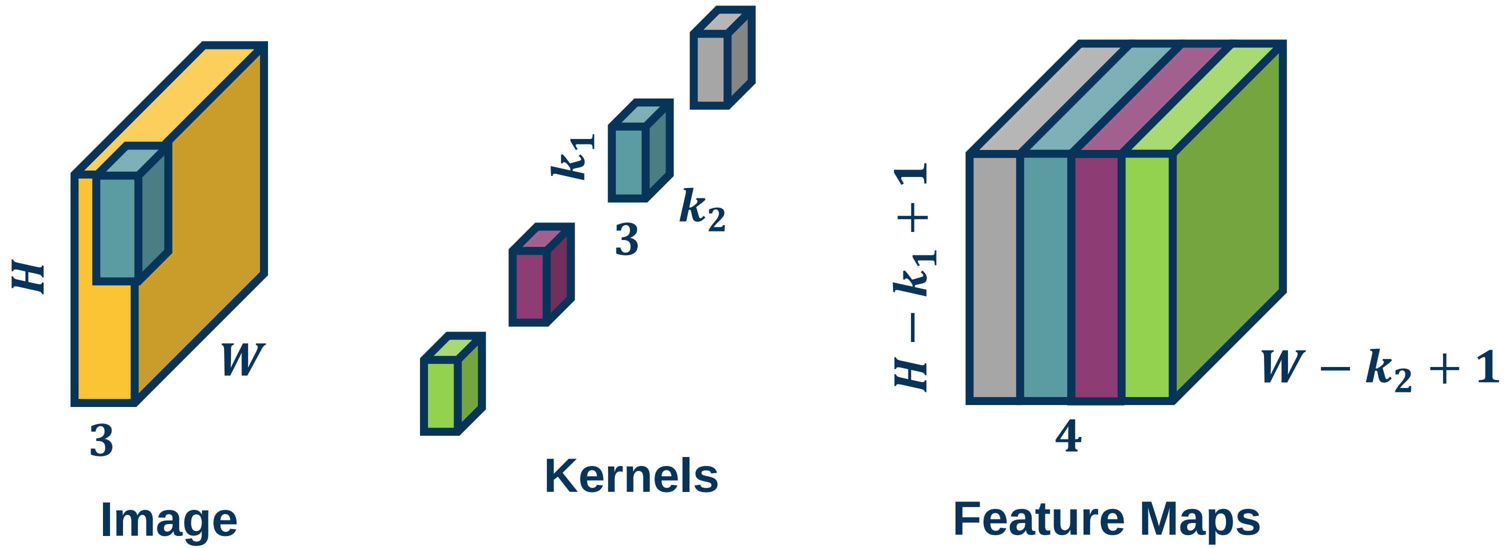
Similar to before, we perform **element-wise multiplication** between kernel and image patch, summing them up **(dot product)**

- Except with $k_1 * k_2 * 3$ values

We can have **multiple kernels per layer**

- We stack the feature maps together at the output

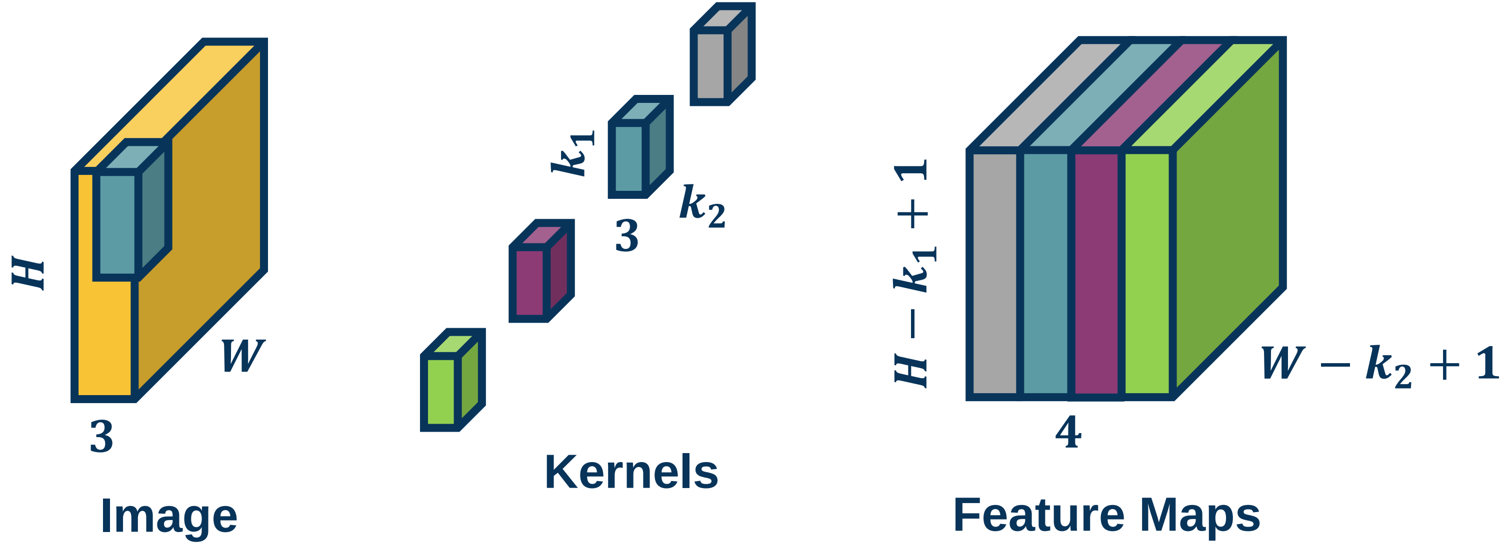
Number of channels in output is equal to *number of kernels*



Multiple Kernels

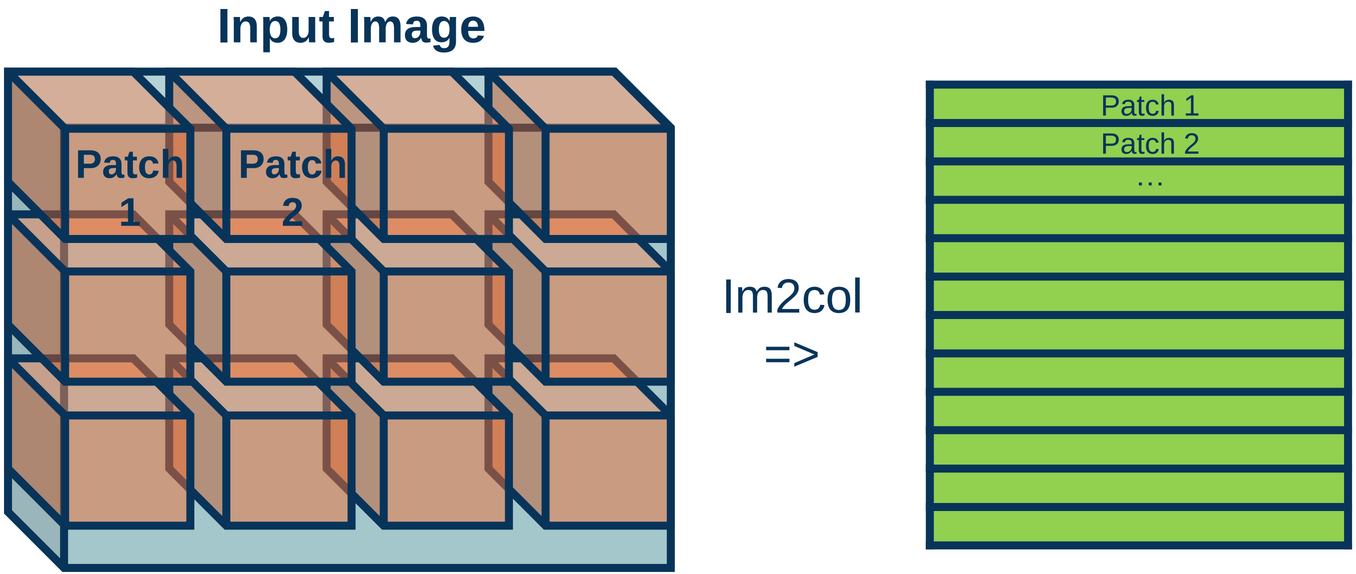
Number of parameters with N filters is: $\boldsymbol{N} * (\boldsymbol{k_1} * \boldsymbol{k_2} * \mathbf{3} + \mathbf{1})$

- Example:
  $\boldsymbol{k_1} = \mathbf{3}, \boldsymbol{k_2} = \mathbf{3},\ \boldsymbol{N} = \mathbf{4}\ \boldsymbol{input\ channels} = \mathbf{3}$, then $(\mathbf{3} * \mathbf{3} * \mathbf{3} + \mathbf{1}) * \mathbf{4} =$112



Just as before, in practice we can **vectorize** this operation

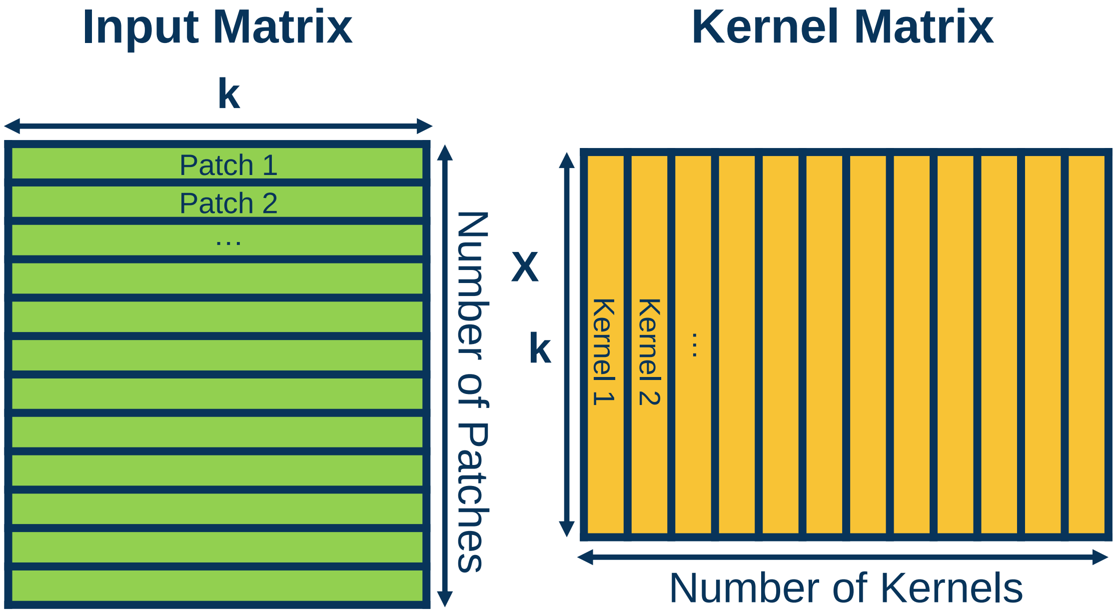
- **Step 1:** Lay out image patches in vector form (note can overlap!)



Adapted from: https://petewarden.com/2015/04/20/why-gemm-is-at-the-heart-of-deep-learning/

Just as before, in practice we can **vectorize** this operation

- **Step 2**: Multiple patches by kernels



Adapted from: https://petewarden.com/2015/04/20/why-gemm-is-at-the-heart-of-deep-learning/

Backwards Pass for Convolution Layer

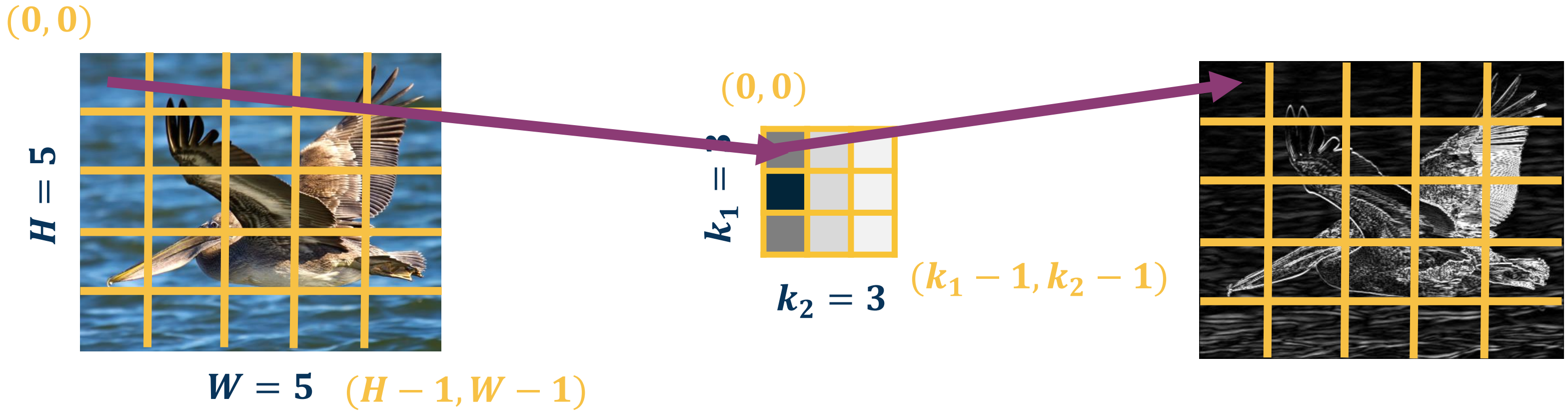
It is instructive to calculate **the backwards pass** of a convolution layer

- Similar to fully connected layer, will be **simple vectorized linear algebra operation**!
- We will see a **duality** between cross-correlation and convolution

$$K = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

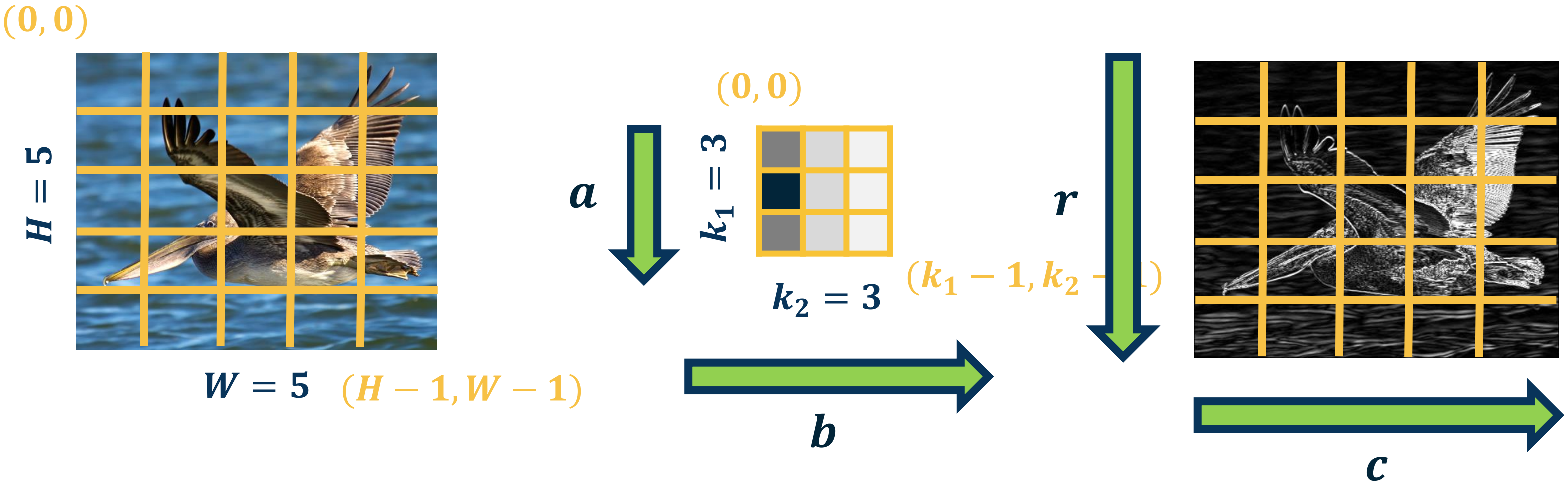
$$K' = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$y(r,c) = (x * k)(r,c) = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} x(r+a, c+b)\, k(a,b)$$



Recap: Cross-Correlation

$$y(r,c) = (x * k)(r,c) = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} x(r+a, c+b)\, k(a,b)$$



Some simplification: 1 channel input, 1 kernel (channel output), padding (here 2 pixels on right/bottom) to make output the same size

$$y(r,c) = (x * k)(r,c) = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} x(r+a, c+b)\, k(a,b)$$

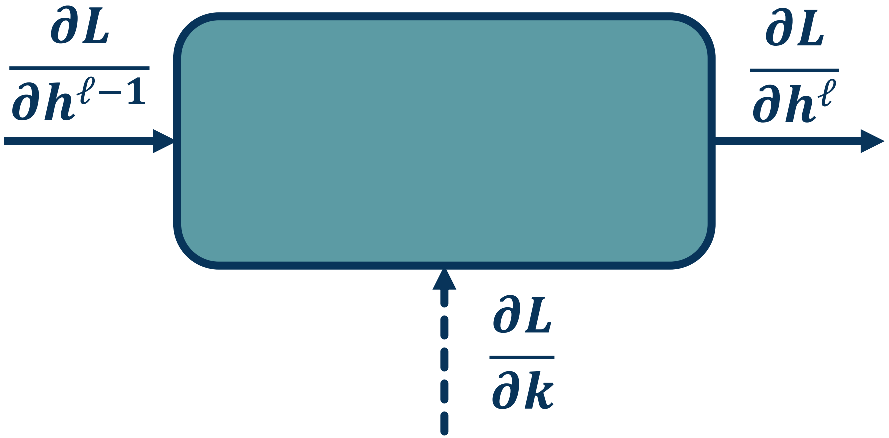
$$|y| = H \times W$$

$\dfrac{\partial L}{\partial y}$? Assume size $H \times W$ (add padding, change convention a bit for convenience)

$\dfrac{\partial L}{\partial y(r,c)}$ to access element

Gradient Terms and Notation

Georgia Tech



$$\frac{\partial L}{\partial h^{\ell-1}} = \frac{\partial L}{\partial h^{\ell}} \frac{\partial h^{\ell}}{\partial h^{\ell-1}}$$

Gradient for passing back

$$\frac{\partial L}{\partial k} = \frac{\partial L}{\partial h^{\ell}} \frac{\partial h^{\ell}}{\partial k}$$

Gradient for weight update

(weights = k, i.e. kernel values)

Backpropagation Chain Rule

Gradient for Convolution Layer

Georgia Tech

What a Kernel Pixel Affects at Output

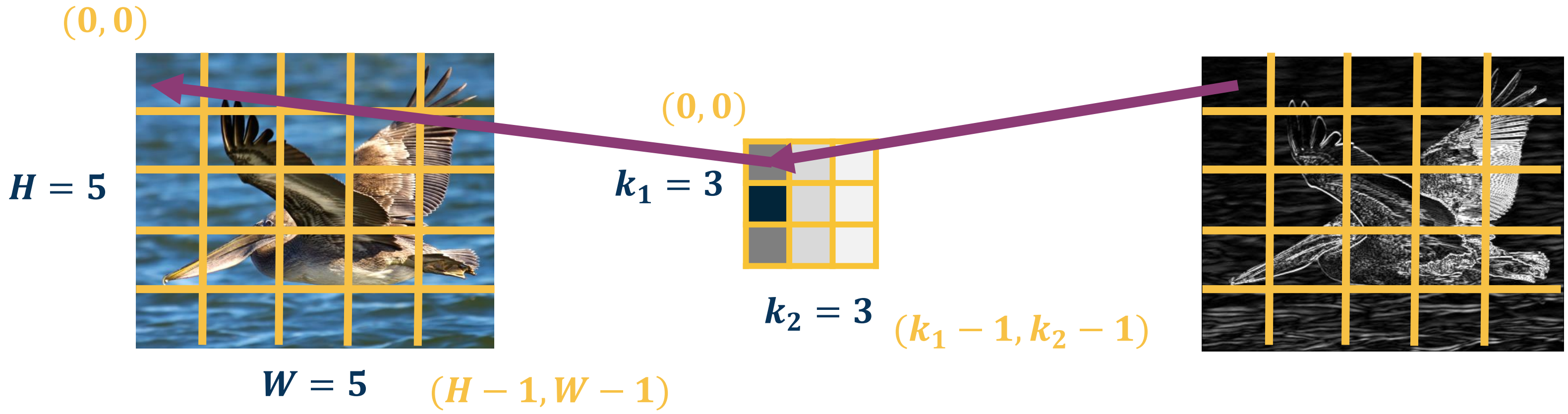
$$\frac{\partial L}{\partial k} = \frac{\partial L}{\partial h^{\ell}} \frac{\partial h^{\ell}}{\partial k}$$

Gradient for weight update

Calculate one pixel at a time $\frac{\partial L}{\partial k(a,b)}$

What does this weight affect at the output?

Everything!



Chain Rule over all Output Pixels

Need to incorporate all upstream gradients:

$$\left\{\frac{\partial L}{\partial y(0,0)}, \frac{\partial L}{\partial y(0,1)}, \ldots, \frac{\partial L}{\partial y(H,W)}\right\}$$

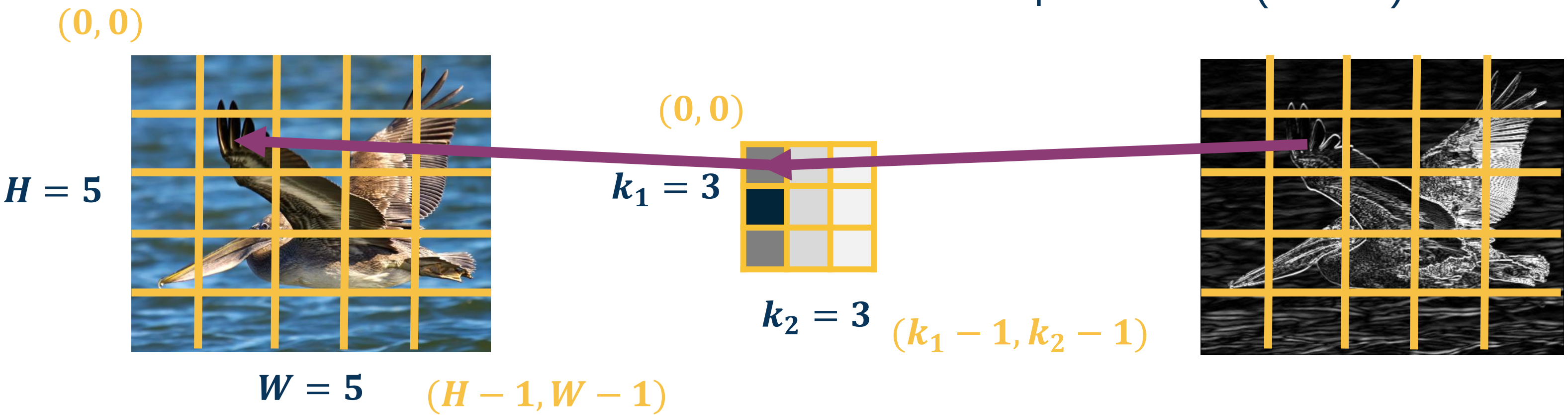
Chain Rule:

$$\frac{\partial L}{\partial k(a,b)} = \sum_{r=0}^{H-1} \sum_{c=0}^{W-1} \frac{\partial L}{\partial y(r,c)} \frac{\partial y(r,c)}{\partial k(a,b)}$$

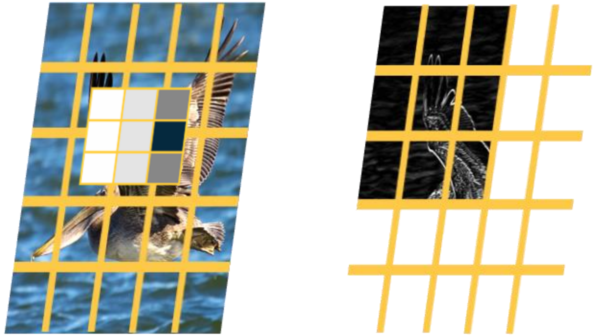
Sum over all output pixels

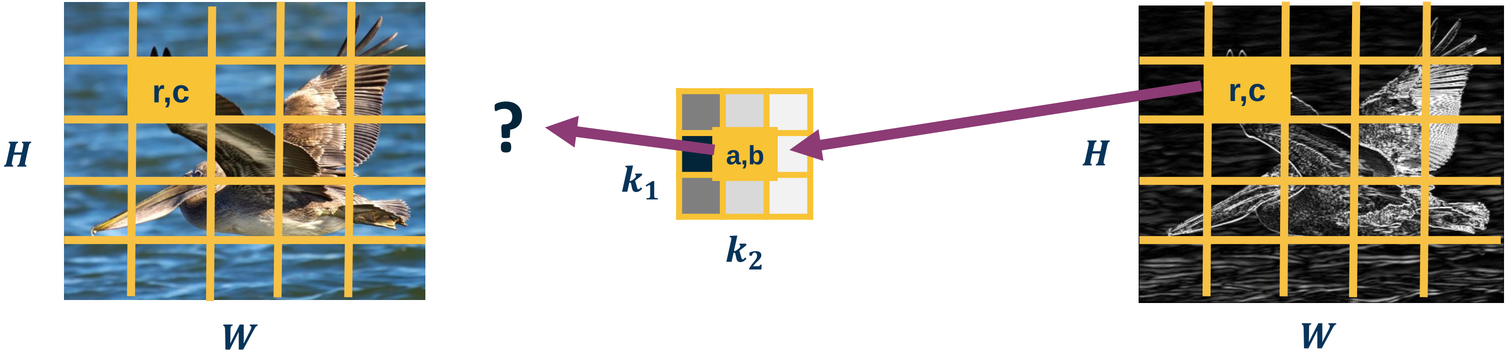
Upstream gradient (known)

We will compute



$$\frac{\partial y(r,c)}{\partial k(a,b)} = ?$$





Chain Rule over all Output Pixels

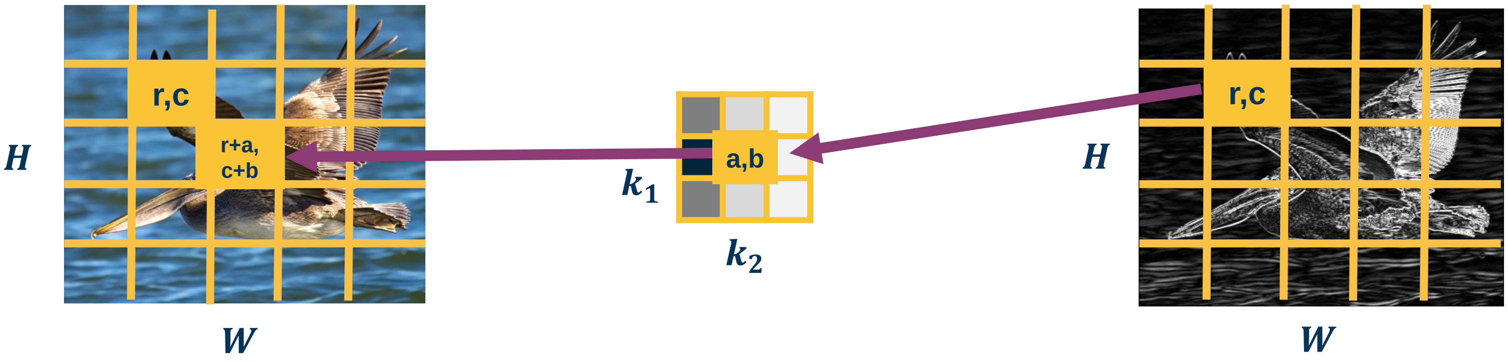
$$\frac{\partial y(r,c)}{\partial k(a,b)} = x(r+a, c+b)$$

$$\frac{\partial L}{\partial k(a,b)} = \sum_{r=0}^{H-1} \sum_{c=0}^{W-1} \frac{\partial L}{\partial y(r,c)} x(r+a, c+b)$$

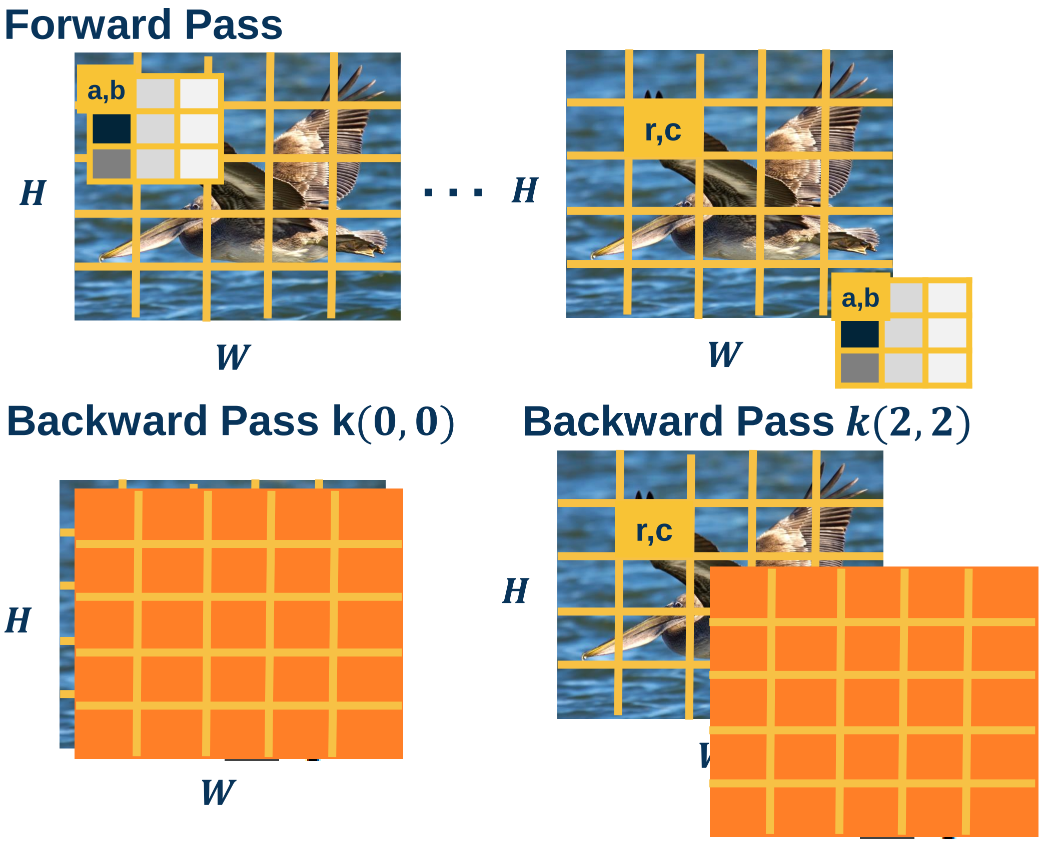
Does this look familiar?

Cross-correlation between upstream gradient and input!

(until $k_1 \times k_2$ output)



Gradients and Cross-Correlation



Does this look familiar?

Cross-correlation between upstream gradient and input!

(until $k_1 \times k_2$ output)

Forward and Backward Duality

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} \frac{\partial y}{\partial x}$$

Gradient for input (to pass to prior layer)

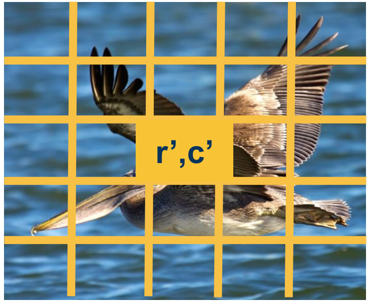
Calculate one pixel at a time $\frac{\partial L}{\partial x(r',c')}$

What does this input pixel affect at the output?

Neighborhood around it (where part of the kernel touches it)

$(0,0)$

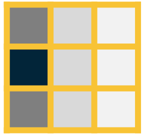
$H = 5$



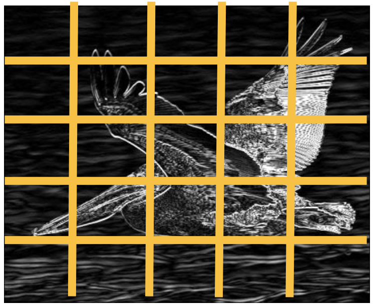
$W = 5$ $(H-1, W-1)$

$(0,0)$

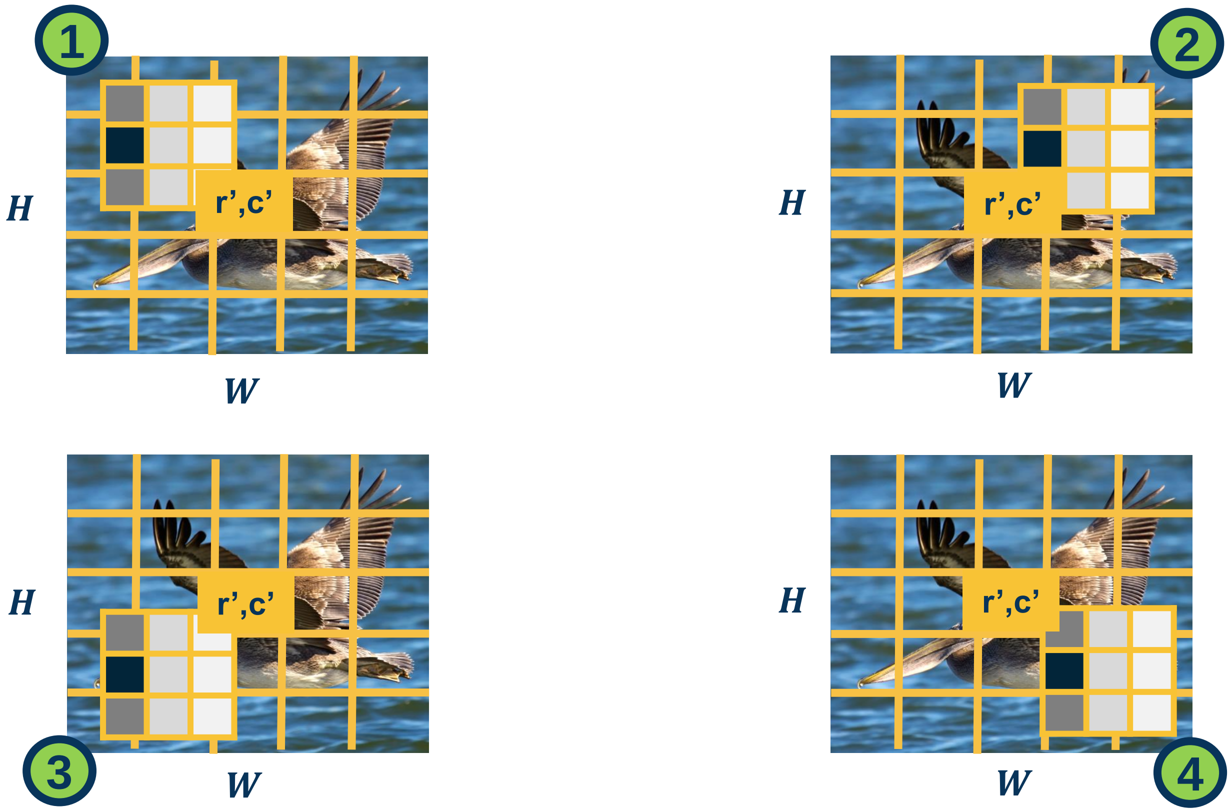
$k_1 = 3$



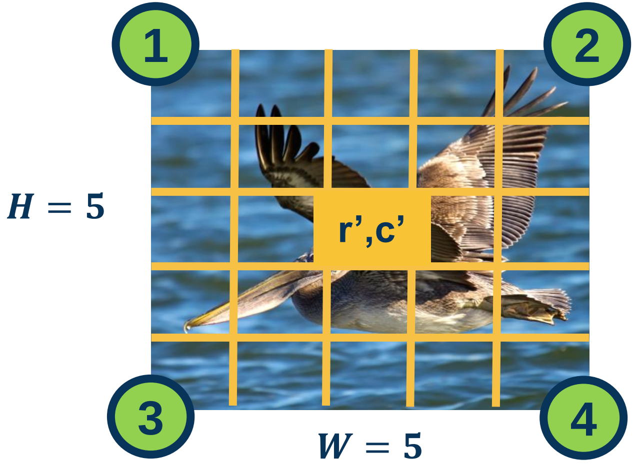
$k_2 = 3$ $(k_1 - 1, k_2 - 1)$

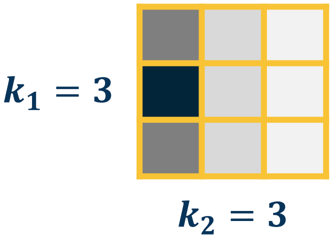


What an Input Pixel Affects at Output

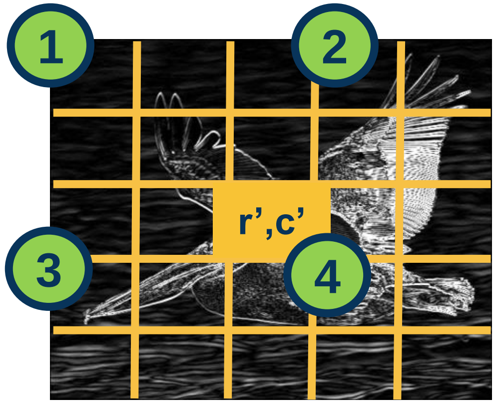


Extents of Kernel Touching the Pixel





$$(r' - k_1 + 1,\ c' - k_2 + 1)$$



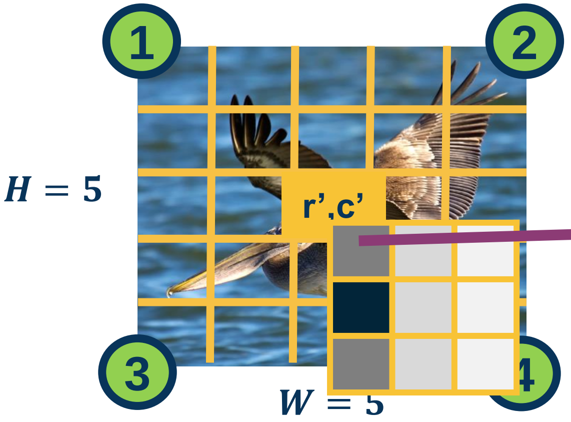
This is where the corresponding locations are for the **output**

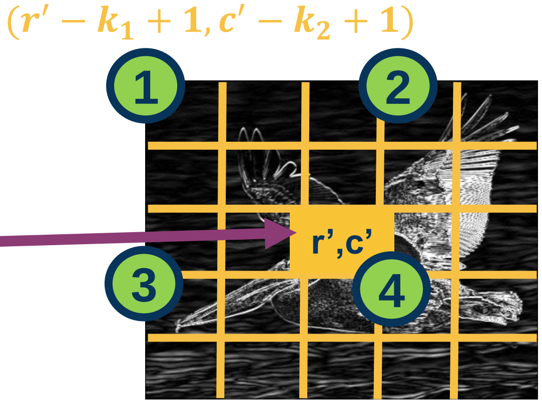
Extents at the Output

Chain rule for affected pixels (sum gradients):

$$\frac{\partial L}{\partial x(r',c')} = \sum_{Pixels\ p} \frac{\partial L}{\partial y(p)} \frac{\partial y(p)}{\partial x(r',c')}$$

$$\frac{\partial L}{\partial x(r',c')} = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(?,?)} \frac{\partial y(?,?)}{\partial x(r',c')}$$





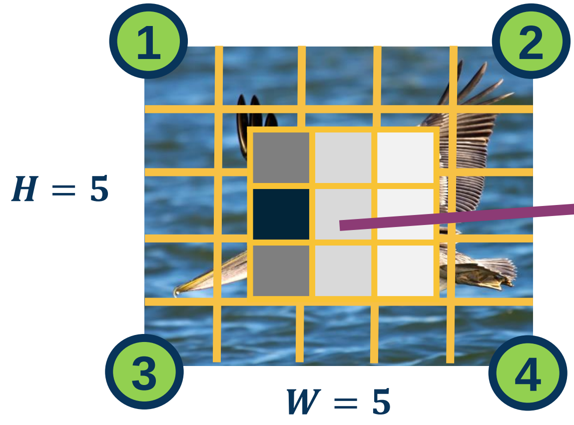
Summing Gradient Contributions

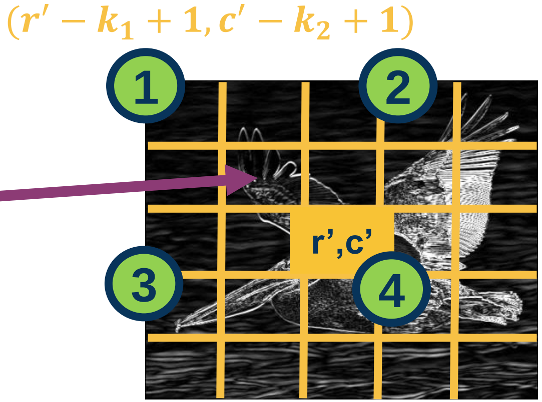
Chain rule for affected pixels (sum gradients):

$$\frac{\partial L}{\partial x(r',c')} = \sum_{Pixels\ p} \frac{\partial L}{\partial y(p)} \frac{\partial y(p)}{\partial x(r',c')}$$

Let's derive it analytically this time (as opposed to visually)

$$\frac{\partial L}{\partial x(r',c')} = \sum_{a=0}^{k_1-1} \sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(r'-a,c'-b)} \frac{\partial y(r'-a,c'-b)}{\partial x(r',c')}$$





Definition of cross-correlation (use a', b' to distinguish from prior variables):

$$\boldsymbol{y}(\boldsymbol{r}', \boldsymbol{c}') = (\boldsymbol{x} * \boldsymbol{k})(\boldsymbol{r}', \boldsymbol{c}') = \sum_{\boldsymbol{a}'=0}^{\boldsymbol{k}_1-1} \sum_{\boldsymbol{b}'=0}^{\boldsymbol{k}_2-1} \boldsymbol{x}(\boldsymbol{r}' + \boldsymbol{a}', \boldsymbol{c}' + \boldsymbol{b}')\, \boldsymbol{k}(\boldsymbol{a}', \boldsymbol{b}')$$

Plug in what we actually wanted :

$$\boldsymbol{y}(\boldsymbol{r}' - \boldsymbol{a}, \boldsymbol{c}' - \boldsymbol{b}) = (\boldsymbol{x} * \boldsymbol{k})(\boldsymbol{r}', \boldsymbol{c}') = \sum_{\boldsymbol{a}'=0}^{\boldsymbol{k}_1-1} \sum_{\boldsymbol{b}'=0}^{\boldsymbol{k}_2-1} \boldsymbol{x}(\boldsymbol{r}' - \boldsymbol{a} + \boldsymbol{a}', \boldsymbol{c}' - \boldsymbol{b} + \boldsymbol{b}')\, \boldsymbol{k}(\boldsymbol{a}', \boldsymbol{b}')$$

What is $\dfrac{\partial \boldsymbol{y}(\boldsymbol{r}' - \boldsymbol{a}, \boldsymbol{c}' - \boldsymbol{b})}{\partial \boldsymbol{x}(\boldsymbol{r}', \boldsymbol{c}')} = \mathbf{k}(\boldsymbol{a}, \boldsymbol{b})$

(we want term with $\boldsymbol{x}(\boldsymbol{r}', \boldsymbol{c}')$ in it;
this happens when $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$)

Plugging in to earlier equation:

$$\frac{\partial L}{\partial x(r',c')} = \sum_{a=0}^{k_1-1}\sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(r'-a,c'-b)} \frac{\partial y(r'-a,c'-b)}{\partial x(r',c')}$$

$$= \sum_{a=0}^{k_1-1}\sum_{b=0}^{k_2-1} \frac{\partial L}{\partial y(r'-a,c'-b)} k(a,b)$$

Does this look familiar?

Convolution between upstream gradient and kernel!

(can implement by flipping kernel and cross- correlation)

Again, all operations can be implemented via matrix multiplications (same as FC layer)!

- Convolutions are mathematical descriptions of striding linear operation

- In practice, we implement **cross-correlation neural networks!** (still called convolutional neural networks due to history)
  - Can connect to convolutions via duality (flipping kernel)
  - Convolution formulation has mathematical properties explored in ECE

- Duality for forwards and backwards:
  - Forward: Cross-correlation
  - Backwards w.r.t. K: Cross-correlation b/w upstream gradient and input
  - Backwards w.r.t. X: Convolution b/w upstream gradient and kernel
    - In practice implement via cross-correlation and flipped kernel

- All operations still implemented via efficient linear algebra (e.g. matrix-matrix multiplication)

Summary