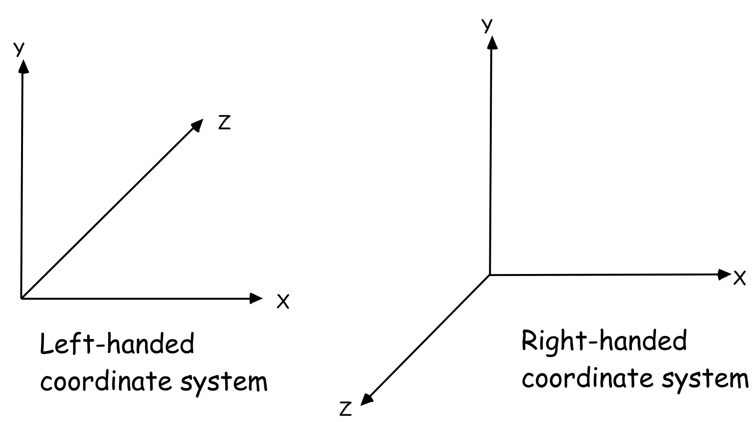
3-D Mathematical Preliminaries

3D Coordinate Systems



3-D Vectors

Have length and direction

$$\mathbf{V} = [x_v, y_v, z_v]$$

Length is given by the Euclidean Norm

$$||\mathbf{V}|| = (x_v^2 + y_v^2 + z_v^2)$$

Dot Product $\mathbf{V} \bullet \mathbf{U} = [x_v, y_v, z_v] \bullet [x_u, y_u, z_u]$

$$= x_v x_u + y_v y_u + z_v z_u$$

$$= ||\mathbf{V}||\ ||\mathbf{U}|| \cos \beta$$

Cross Product $\mathbf{V} \times \mathbf{U} = [v_y u_z - v_z u_y, -v_x u_z + v_z u_x, v_x u_y - v_y u_x]$

$$\mathbf{V} \times \mathbf{U} = -(\mathbf{U} \times \mathbf{V})$$

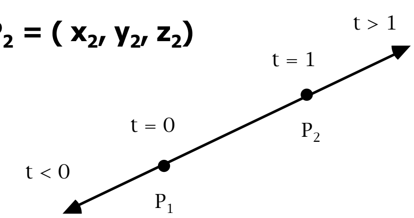
Parametric Definition of a Line

Given two points: $\mathbf{P_1} = (x_1, y_1, z_1)$, $\mathbf{P_2} = (x_2, y_2, z_2)$

$$x = x_1 + t\,(x_2 - x_1)$$

$$y = y_1 + t\,(y_2 - y_1)$$

$$z = z_1 + t\,(z_2 - z_1)$$



Given a point $\mathbf{P_1}$ and a vector $\mathbf{V} = [x_v, y_v, z_v]$

$$x = x_1 + t\,x_v, \quad y = y_1 + t\,y_v, \quad z = z_1 + t\,z_v$$

Short form: $\mathbf{L = P_1 + t[P_2 - P_1]}$ **or** $\mathbf{L = P_1 + Vt}$

Equation of a plane:

$Ax + By + Cz + D = 0$

Normalized Form: $A'x + B'y + C'z + D' = 0$

where $A' = A/d$, $B' = B/d$, $C' = C/d$, $D' = D/d$

$d = (A^2 + B^2 + C^2)$

Distance between a point and the plane is given by

$A'x + B'y + C'z + D'$ (sign indicates which side)

[A, B, C] is the normal vector

Proof: Given P_1 and P_2 in the plane, $[P_2 - P_1]$ is in the plane and

$$[A,B,C] \bullet [P_2 - P_1] = (Ax_2 + By_2 + Cz_2) - (Ax_1 + By_1 + Cz_1)$$
$$= (-D) - (-D)$$
$$= 0$$

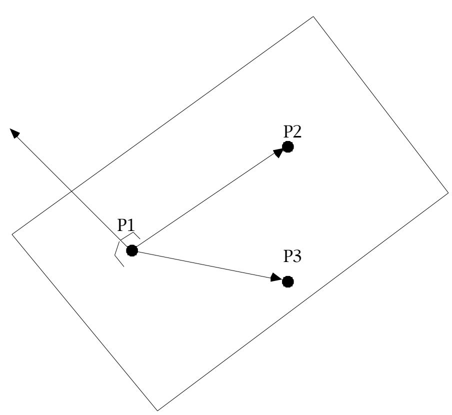
Derivation of Plane Equation

To derive equation of the plane given three points: **P_1, P_2, P_3**

$[P_3 - P_1] \times [P_2 - P_1] = N$, orthogonal vector

Given a point **$P = (x,y,z)$**

$N \bullet [P - P_1] = 0$ if **P** is in the plane.



Affine Transformations

- Linear transformations (rotation, scale, shear,…) plus translation
  - Represented as matrices
- Objects defined in local coordinates
  - Transformed to other reference frames
    - E.g., world coordinates
- Transform objects by xforming vertices

Homogeneous Coordinates

- Represent transformations as matrices
  - easier to manipluate and use
- How to represent translation?
  - Use 3 x 3 matrices for 2D xform, 4x4 for 3D
- Represent points as 1x3 or 1x4 vectors
  - point P = (x, y, 1) or (x, y, z, 1)

Homogeneous 3-D Coordinates

(**T** is any transformation, **P** is any point)

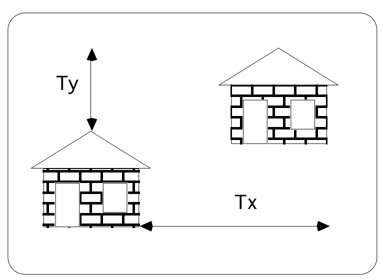
TP = T(x,y,z,1) = (x', y', z',w)

Homogenize the result:

P_h = (x'/w, y'/w, z'/w,1)

Translations

- Translation = moving an object
- Translate object
  - translate each vertex
- Translate point
  - add translation (tx, ty) to vertex (x1, y1)



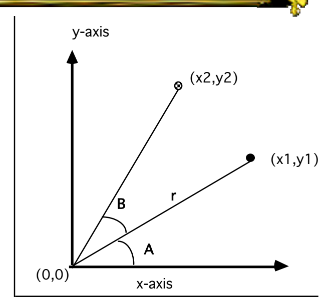
$x_2 = x_2 + t_x$ $y_2 = y_1 + t_y$

Translation

(1 0 0 t_x) (x)
(0 1 0 t_y) (y)
(0 0 1 t_z) (z)
(0 0 0 1) (1)
T P

TP = (x + t_x, y + t_y, z + t_z)

Rotation About the Origin



sin (A + B) = y2/r
cos (A + B) = x2/r
sin A = y1/r, cos A = x1/r

From the double angle formulas

sin (A + B) = sinA cosB + cosA sinB
 y2/r = (y1/r)cosB + (x1/r)sinB
y2 = x1 sinB + y1 cosB

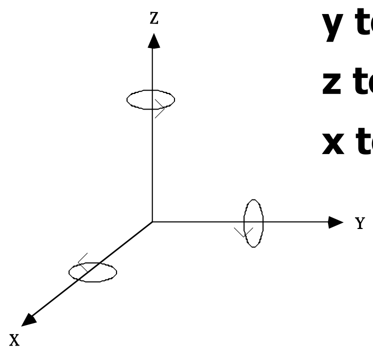
Similarly

x2 = x1 cosB - y1 sinB

cos ø -sin ø 0
 Rø = sin ø cos ø 0
0 0 1

3D Rotations

Axis of rotation is	Direction of positive rotation is
x	**y to z**
y	**z to x**
z	**x to y**

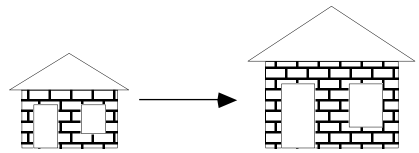


3D Rotations

About the z axis	R_z(ß) P =(cosß	-sinß	0	0)		(x)
	(sinß	cosß	0	0)		(y)
	(0	0	1	0)		(z)
	(0	0	0	1)		(1)
About the x axis	R_x(ß) P =(1	0	0	0)		(x)
	(0	cosß	-sinß	0)		(y)
	(0	sinß	cosß	0)		(z)
	(0	0	0	1)		(1)
About the y axis	R_y(ß) P =(cosß	0	sinß	0)		(x)
	(0	1	0	0)		(y)
	(-sinß	0	cosß	0)		(z)
	(0	0	0	1)		(1)

Scaling

- Scaling = changing the size of an object
- Scale object
  - scale each vertex



$x_2 = s_x x_1$ $y_2 = s_y y_1$

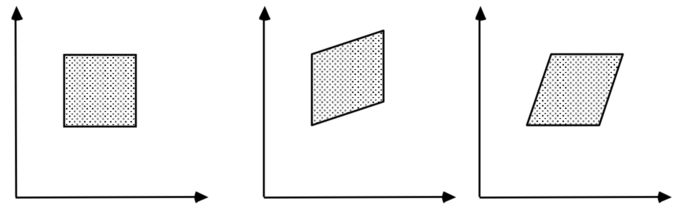
- Scale point
  - multiply scale factor (sx, sy) by vertex (x1, y1)

Scale

$$\underbrace{\begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{S} \underbrace{\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}}_{P}$$

$$SP = (s_x x,\ s_y y,\ s_z z)$$

Shears



Original Data   y Shear   x Shear

e.g., GRAPHICS → x shear → *GRAPHICS*

Shears

$$SH_{xy}P = \begin{pmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$SH_{xy}P = (x+sh_x z,\ y+sh_y z,\ s_z z)$$

Composite Transformations

if

P′=M_1P and **P″ = M_2P′**

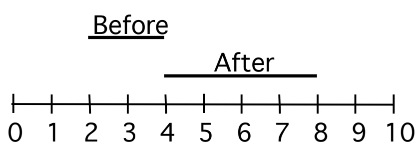
then

M_3 = M_2M_1 and **P″ = M_3P**

NOTE: in general, **$M_2M_1 \neq M_1M_2$**

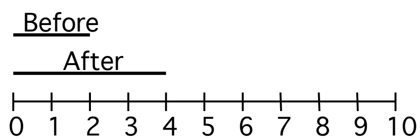
Composite Transformations

- Problem:
  - scale transformation moves the object being scaled
- i.e. scale the line [(2, 1), (4,1)] by 2x



Composite Transformations

- Notice: scale line [(0, 1), (2,1)] by 2x
  □ left end does not move



(0,0) is a ***fixed point*** for the scaling transformation

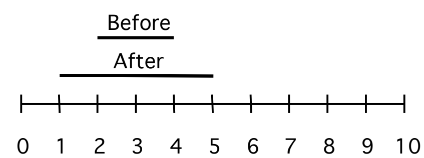
Use composite transformations to create scale transformations with different fixed points

Fixed Point Scaling

- Scale by 2 with fixed point = (2,1)
  - Translate the point (2,1) to the origin
  - Scale by 2
  - Translate origin to point (2,1)

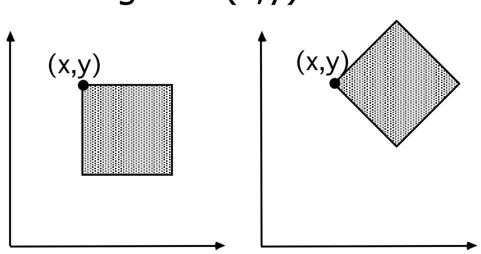
More Fixed Point Scaling

- Scale by 2 with fixed point = (3,1)
  - Translate the point (3,1) to the origin
  - Scale by 2
  - Translate origin to point (3,1)



Rotation About a Fixed Point

- Rotation Of ø Degrees About Point (x,y)
  - Translate (x,y) to origin
  - Rotate by ø
  - Translate origin to (x,y)



Rotation About An Arbitrary Axis

1. Translate one end of the axis to the origin

$U = [P_2 - P_1] = [\ u_1, u_2, u_3]$

Some useful values:

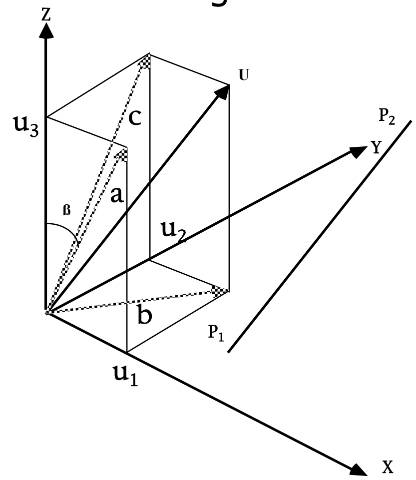
$a = \quad (u_1^2 + u_3^2\ )$

$b = \quad (u_1^2 + u_2^2)$

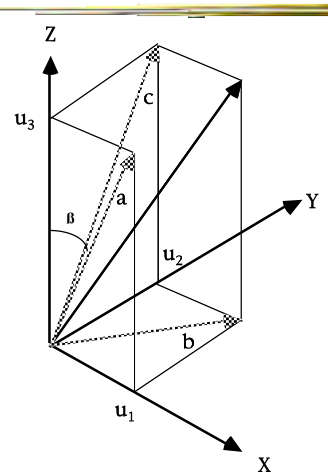
$c = \quad (u_2^2 + u_3^2)$

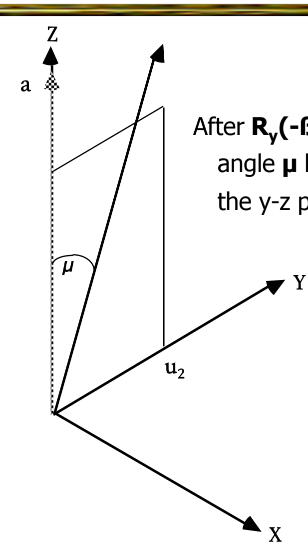
$\cos ß = \ u_3/a$

$\sin ß = \ u_1/a$



2. Rotate -ß degrees about the y-axis





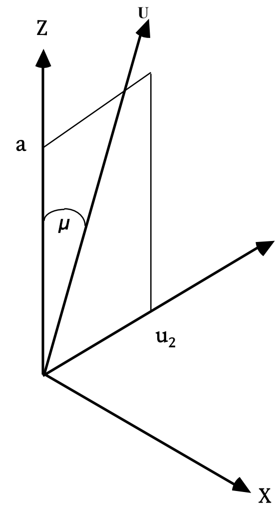
After $\mathbf{R_y(-ß)}$,
angle **μ** lies in
the y-z plane

3. Rotate -μ degrees about the x-axis

R_x (μ)

cos μ = a/||u||

sin μ = u_2/||u||

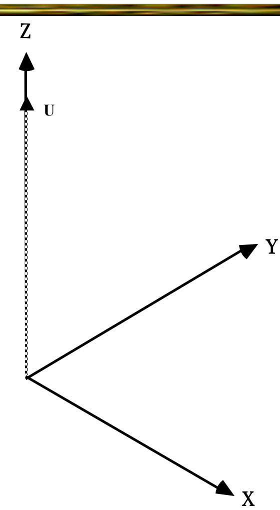


4. Rotate R degrees about the z-axis

U is aligned with the z-axis

Apply the original rotation, R

5. Apply the inverses of the transformations in reverse order.



Rotation About an Arbitrary Axis

- $T^{-1}\ R_y(\beta)\ R_x(-\mu)$ R $R_x(\mu)\ R_y(-\beta)\ T$

Alternate view of the Rotation Matrix

Given P_1, P_2, P_3

P_1P_2 is direction, P_1P_3 is "up"

$$R_{cc} = \begin{matrix} (r_{1x} & r_{2x} & r_{3x} & 0) \\ (r_{1y} & r_{2y} & r_{3y} & 0) \\ (r_{1z} & r_{2z} & r_{3z} & 0) \\ (0 & 0 & 0 & 1) \end{matrix}$$

Alternate view of the Rotation Matrix

Z axis rotates to be aligned with P_1P_2

$R_z = [\mathbf{r}_{3x}, \mathbf{r}_{3y}, \mathbf{r}_{3z}]$ = normalized P_1P_2

X axis rotates to be normal to P_1, P_2, P_3 plane

$R_x = [\mathbf{r}_{1x}, \mathbf{r}_{1y}, \mathbf{r}_{1z}]$ = normalized $P_1P_3 \times P_1P_2$

Y axis rotates to be normal to $R_x R_z$ plane

$R_y = [\mathbf{r}_{2x}, \mathbf{r}_{2y}, \mathbf{r}_{2z}]$ = normalized $R_z \times R_x$

Transformations as a Change of Coordinate System

- Objects modelled in local coordinates
  - Xforms that move objects into world coordingates are called *modeling xforms*
- If xform **M** takes points from $\mathbf{CS_1}$ to $\mathbf{CS_2}$
  - $\mathbf{M^{-1}}$ takes origin of $\mathbf{CS_1}$ to $\mathbf{CS_2}$