Curves and Surfaces

Curves

- Implicit/Explicit
- Piecewise
- Parametric cubic curves
  - Basic ideas
  - Hermite
  - Bezier
  - B-Splines

Explicit/Implicit

- Explicit Functions: **$y = f(x)$** (e.g., **$y = 2x$**)
  - Only one value of y for each x
  - Not rotationally invarient
  - Difficult to represent a slope of infinity
- Implicit Equations: **$f(x,y) = 0$** (e.g., **$x^2+y^2-r^2 = 0$**)
  - Need constraints to model part of a curve
  - Joining curves together smoothly is difficult

Piecewise Parametric Curves

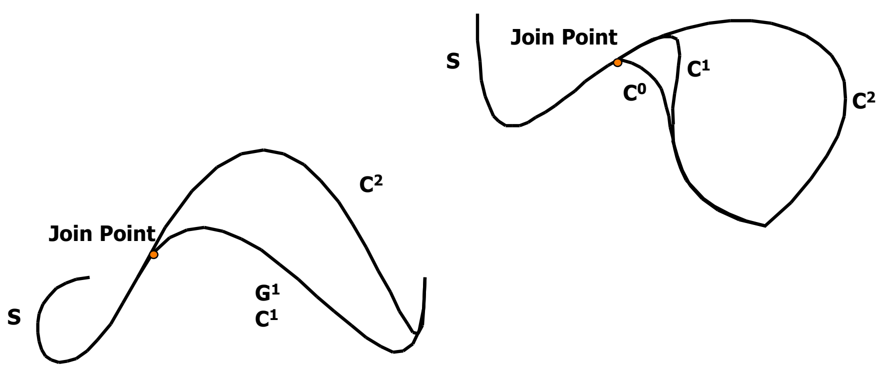
- Piecewise curves
  - Use multiple simple curves to model a complex curve in pieces
- Parametric Equations:

  for $0 \le t \le 1$, $x = f(t)$, $y = g(t)$ (e.g., **$x = 3 + 4t$, $y = 2 - 2t$**)
  - Easy to join curve segments smoothly
  - Slopes are parametric tangent vectors

Piecewise Curves: Continuity

- G^0: geometric continuity
  - Curve segments join together
- G^1: geometric continuity
  - Tangent vectors equal directions at join point
- C^1: parametric continuity
  - Tangents have equal magnitude and direction
- C^2: parametric continuity
  - Direction and magnitude of n^{th} derivative equal

Continuity Examples



Parametric Cubic Curves

- Use cubic curves

$$x = a_x t^3 + b_x t^2 + c_x t + d_x$$
$$y = a_y t^3 + b_y t^2 + c_y t + d_y$$
$$z = a_z t^3 + b_z t^2 + c_z t + d_z$$

- Notation: $\mathbf{Q(t) = T\ C}$, where

$$\mathbf{T} = (t^3 \quad t^2 \quad t \quad 1) \qquad \mathbf{C} = \begin{matrix} (a_x & a_y & a_z) \\ (b_x & b_y & b_z) \\ (c_x & c_y & c_z) \\ (d_x & d_y & d_z) \end{matrix}$$

Why Cubic?

- Lower: inflexible
- Higher: hard to control, expensive

- 4 coefficients 4 unknowns needed
  - e.g. endpoints, tangents, continuity
  - What they are determine the kind of curve

Types of Curves

- Hermite
  - 2 endpoints, 2 endpoint tangents
- Bezier
  - 2 endpoints, 2 other points defining tangents
- B-Splines
  - 4+ control points, C^n continuous
  - approximates control points

Types of Curves

- Non-Uniform B-Splines
  - Control points (knots) can be repeated
    - Curve can be forced through control points
    - Sharp corners can be created
- Rational curves
  - 3D curves modeled in 4-space
    - Control points are **(xw, yw, zw, w)**
  - Weight can pull curve toward control points
  - Non-affine transforms (e.g., projection)

Defining Cubic Polys

$$Q(t) = TC = T\,M\,G$$

$$= \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix}$$

T Matrix **Basis Matrix** **Geometry Matrix**

Defining Cubic Polys

i.e. $x(t) = T\,M\,G_x$

$$= \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} g_{1x} \\ g_{2x} \\ g_{3x} \\ g_{4x} \end{pmatrix}$$

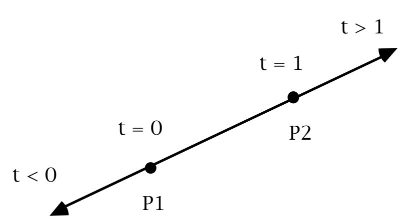
Similarly for **y(t)** and **z(t)**

Defining Cubic Polys

- Basis matrix defines type of cubic
- **TM** □ 4 cubic polynomials
  - *blending functions*
  - Defined to achieve desired props for G
- Curve is weighted sum of elements of G
  - Weights are cubics in **t**

2D example

- Recall **Q(t) = T M G**
- Lines:
  - $\mathbf{x(t) = (1-t)\, g_{1x} + (t)\, g_{2x}}$
- Blending functions
  - **-t + 1, t**
- **M = [-1 1]**
  [1 0]



Aside: Derivatives

- Parametric Curves

  $x = a_x t^3 + b_x t^2 + c_x t + d_x$

  $y = a_y t^3 + b_y t^2 + c_y t + d_y$

  $z = a_z t^3 + b_z t^2 + c_z t + d_z$

- Derivatives of Parametric Curves

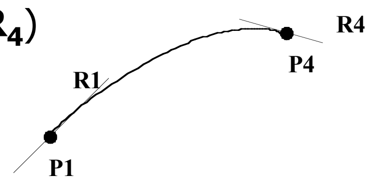
  $dx/dt = 3a_x t^2 + 2b_x t + c_x$

  $dy/dt = 3a_y t^2 + 2b_y t + c_y$

  $dz/dt = 3a_z t^2 + 2b_z t + c_z$

Hermite Curves

- **$Q(t) = T\, M_H\, G_H$**
  - **M_H** is the Hermite Basis Matrix
  - **G_H** is the Hermite Geometry Matrix
- **G_H**: 4 triples that define the curve
  - 2 endpoints (**P_1 P_4**)
  - 2 endpoint tangents (**R_1 R_4**)
- What is **M_H**?



Hermite Basis Matrix M_H

- Given: $\mathbf{P_1}$, $\mathbf{P_4}$ and $\mathbf{R_1}$, $\mathbf{R_4}$

  Recall: $x(t) = TM_HG_{Hx}$, $x'(t) = T' M_HG_{Hx}$
- Thus:

$$x(0) = P_{1x} = [0\ 0\ 0\ 1]\ M_HG_{Hx}$$
$$x(1) = P_{4x} = [1\ 1\ 1\ 1]\ M_HG_{Hx}$$
$$x'(0) = R_{1x} = [0\ 0\ 1\ 0]\ M_HG_{Hx}$$
$$x'(1) = R_{4x} = [3\ 2\ 1\ 0]\ M_HG_{Hx}$$

Hermite Basis Matrix

- But,

$$\begin{bmatrix} P_{1x} \\ P_{4x} \\ R_{1x} \\ R_{4x} \end{bmatrix} = G_{Hx} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} M_HG_{Hx}$$

- Thus, $\mathbf{M_H}$ = inverse of above matrix

$$= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hermite Blending Functions

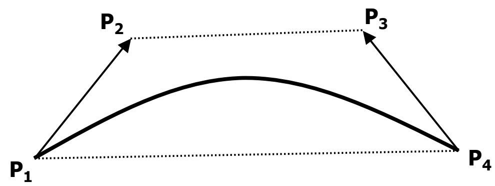
- **$Q(t) = T\, M_H\, G_H$**
  $= (2t^3 - 3t^2 + 1)\, P_1 +$
  $(-2t^3 + 3t^2)\, P_4 +$
  $(t^3 - 2t^2 + t)\, R_1 +$
  $(t^3 - t^2)\, R_4$

Bezier Curves

- **$Q(t) = T\, M_B\, G_B$**
  - **M_B** is the Bezier Basis Matrix
  - **G_B** is the Bezier Geometry Matrix
- **G_B**: 4 control points
  - 2 endpoints (**P_1 P_4**)
  - 2 points defining tangents (**P_2 P_3**), where
    - **$R_1 = 3(P_2 - P_1)$**
    - **$R_4 = 3(P_4 - P_3)$**

Bezier Curves

- Bounded by convex hull of control points



Bezier Basis Matrix

- But,

$$G_B = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \quad G_H = \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} G_B = M_{HB} G_B$$

- Thus, $M_B = M_H M_{HB}$

$$= \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Bezier Blending Functions

- The Bernstein Polynomials
- **$Q(t) = T\ M_B\ G_B$**
  $= (1-t)^3\ P_1 +$
  $3t(1-t)^2\ P_2 +$
  $3t^2(1-t)\ P_3 +$
  $t^3\ P_4$