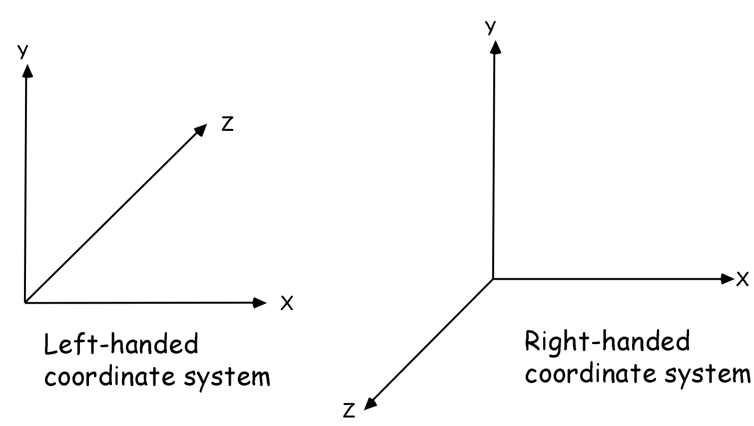
3-D Mathematical Preliminaries

3D Coordinate Systems



3-D Vectors

Have length and direction

$V = [x_v, y_v, z_v]$

Length is given by the Euclidean Norm

$||V|| = \sqrt{(x_v^2 + y_v^2 + z_v^2)}$

Dot Product $V \cdot U = [x_v, y_v, z_v] \cdot [x_u, y_u, z_u]$

$= x_v x_u + y_v y_u + z_v z_u$

$= ||V|| \ ||U|| \cos \beta$

Cross Product $V \times U = [v_y u_z - v_z u_y, \ -v_x u_z + v_z u_x, \ v_x u_y - v_y u_x]$

$V \times U = - (U \times V)$

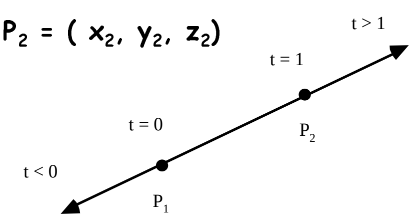
Parametric Definition of a Line

Given two points: $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$

$x = x_1 + t (x_2 - x_1)$

$y = y_1 + t (y_2 - y_1)$

$z = z_1 + t (z_2 - z_1)$



Given a point P_1 and a vector $V = [x_v, y_v, z_v]$

$x = x_1 + t x_v$, $y = y_1 + t y_v$, $z = z_1 + t z_v$

Short form: $L = P_1 + t[P_2 - P_1]$ or $L = P_1 + Vt$

Equation of a plane:

$Ax + By + Cz + D = 0$

Normalized Form: **$A'x + B'y + C'z + D' = 0$**

where **$A' = A/d$, $B' = B/d$, $C' = C/d$, $D' = D/d$**

$d = \sqrt{(A^2 + B^2 + C^2)}$

Distance between a point and the plane is given by

$A'x + B'y + C'z + D'$ (sign indicates which side)

[A, B, C] is the normal vector

Proof: Given P_1 and P_2 in the plane, $[P_2 - P_1]$ is in the plane and

$$[A,B,C] \cdot [P_2 - P_1] = (Ax_2 + By_2 + Cz_2) - (Ax_1 + By_1 + Cz_1)$$
$$= (-D) - (-D)$$
$$= 0$$

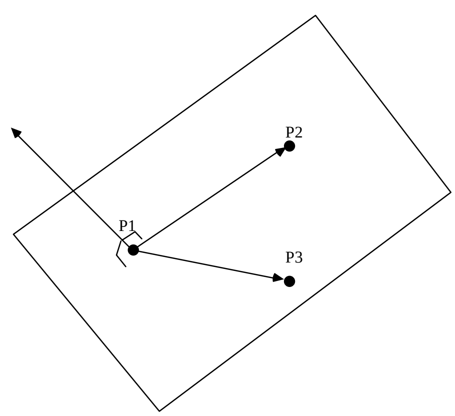
Derivation of Plane Equation

To derive equation of the plane given three points: **P_1, P_2, P_3**

$[P_3 - P_1] \times [P_2 - P_1] = N$, orthogonal vector

Given a point **$P = (x,y,z)$**

$N \cdot [P - P_1] = 0$ if **P** is in the plane.



Homogeneous Coordinates

- Represent transformations as matrices
  - easier to manipluate and use
- Use 3 x 3 matrices for 2D xform
  - needed to represent a translation as a matrix operation
- Represent points as 1 x 3 matrices
  - point P = (x, y, 1)

Homogeneous 3-D Coordinates

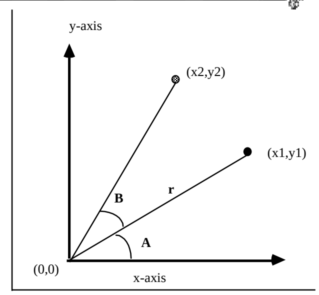
(**T** is any transformation, **P** is any point)

TP = T(x,y,z,1) = (x', y', z',w)

Homogenize the result:

P_h = **(x'/w, y'/w, z'/w,1)**

Rotation About the Origin



sin (A + B) = y2/r

cos (A + B) = x2/r

sin A = y1/r, cos A = x1/r

From the double angle formulas

sin (A + B) = sinA cosB + cosA sinB

∴ y2/r = (y1/r)cosB + (x1/r)sinB

y2 = x1 sinB + y1 cosB

Similarly

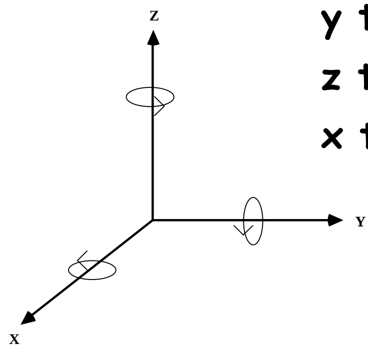
x2 = x1 cosB - y1 sinB

2D Rotation

$$R_ø = \begin{matrix} \cos ø & -\sin ø & 0 \\ \sin ø & \cos ø & 0 \\ 0 & 0 & 1 \end{matrix}$$

3D Rotations

Axis of rotation is	Direction of positive rotation is
x	**y to z**
y	**z to x**
z	**x to y**



3D Rotations

About the z axis

$$R_z(\beta)\,P = \begin{pmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

About the x axis

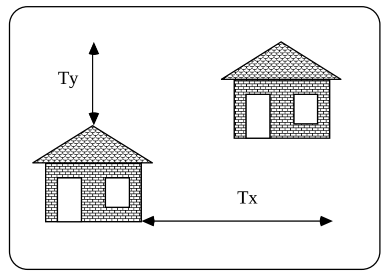
$$R_x(\beta)\,P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta & 0 \\ 0 & \sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

About the y axis

$$R_y(\beta)\,P = \begin{pmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Translations

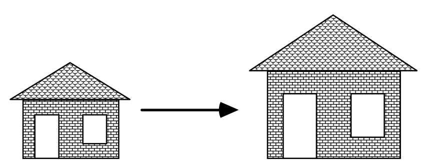
- Translation = moving an object
- Translate object
  - translate each vertex
- Translate point
  - add translation (tx, ty) to vertex (x1, y1)



$x_2 = x_2 + t_x$ $y_2 = y_1 + t_y$

Scaling

- Scaling = changing the size of an object
- Scale object
  - scale each vertex



$x_2 = s_x x_1$ $y_2 = s_y y_1$

- Scale point
  - multiply scale factor (sx, sy) by vertex (x1, y1)

Scale and Translation

	Sx	0	0
S =	0	Sy	0
	0	0	1

	1	0	Tx
T =	0	1	Ty
	0	0	1

Scale

SP = $(s_x x,\ s_y y,\ s_z z)$

(s_x	**0**	**0**	**0)**	**(x)**
(0	**s_y**	**0**	**0)**	**(y)**
(0	**0**	**s_z**	**0)**	**(z)**
(0	**0**	**0**	**1)**	**(1)**

Translation

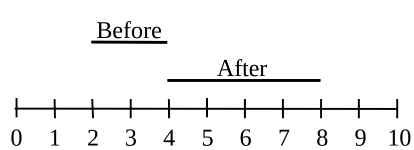
$TP = (x + t_x,\ \ y + t_y,\ \ z + t_z)$

$$\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$\quad T \qquad\qquad\qquad P$$

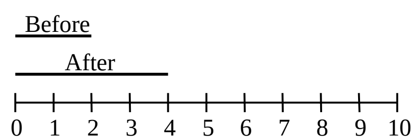
Composite Transformations

- Problem:
  - scale transformation moves the object being scaled
- i.e. scale the line [(2, 1), (4,1)] by 2x



Composite Transforms (cont.)

- Notice: scale line [(0, 1), (2,1)] by 2x
  ⇒ left end does not move

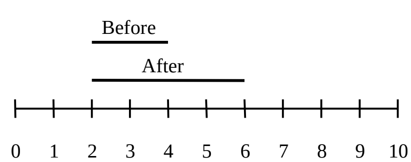


(0,0) is a ***fixed point*** for the scaling transformation

Use composite transformations to create scale transformations with different fixed points

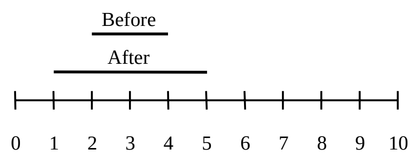
Fixed Point Scaling

- Scale by 2 with fixed point = (2,1)
  - Translate the point (2,1) to the origin
  - Scale by 2
  - Translate origin to point (2,1)



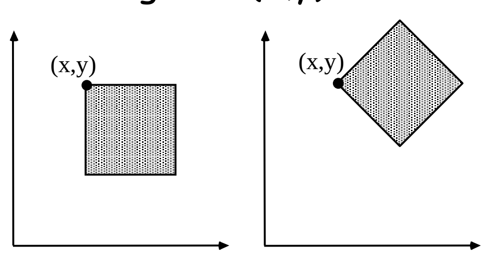
More Fixed Point Scaling

- Scale by 2 with fixed point = (3,1)
  - Translate the point (3,1) to the origin
  - Scale by 2
  - Translate origin to point (3,1)

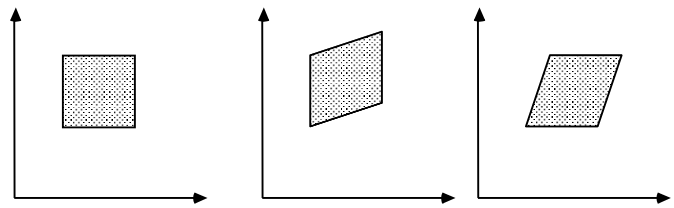


Rotation About a Fixed Point

- Rotation Of ø Degrees About Point (x,y)
  - Translate (x,y) to origin
  - Rotate by ø
  - Translate origin to (x,y)



Shears



Original Data y Shear x Shear

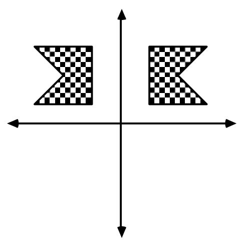
e.g., GRAPHICS ⇒ x shear ⇒ *GRAPHICS*

Shears

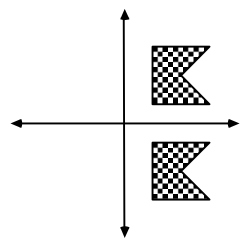
$$SH_{xy}P = \begin{pmatrix} 1 & 0 & sh_x & 0 \\ 0 & 1 & sh_y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Reflections

Y-axis

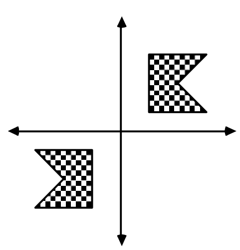


X-axis

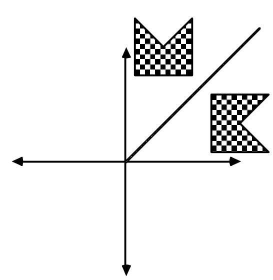


More Reflections

Origin



y=x



Rotation About An Arbitrary Axis

1. Translate one end of the axis to the origin

$U = [P_2 - P_1] = [\ u_1,\ u_2,\ u_3]$

Some useful values:

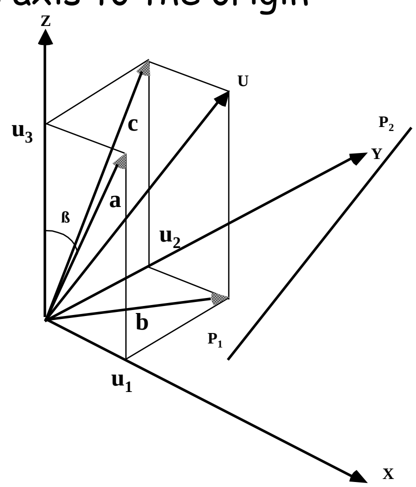
$a = \sqrt{(u_1^2 + u_3^2)}$

$b = \sqrt{(u_1^2 + u_2^2)}$

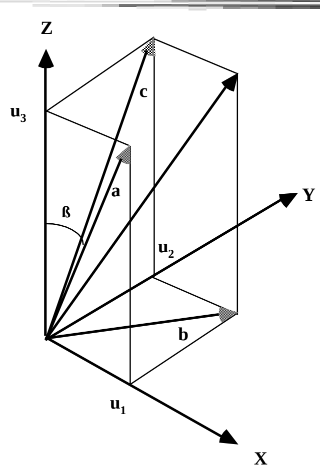
$c = \sqrt{(u_2^2 + u_3^2)}$

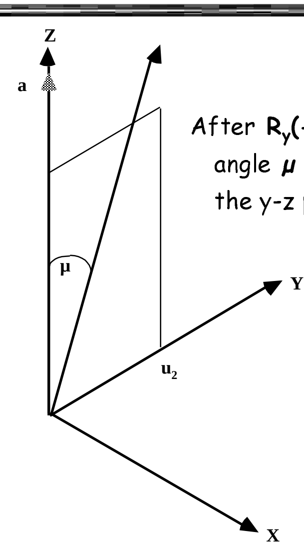
$\cos ß = u_3/a$

$\sin ß = u_1/a$



2. Rotate -ß degrees about the y-axis

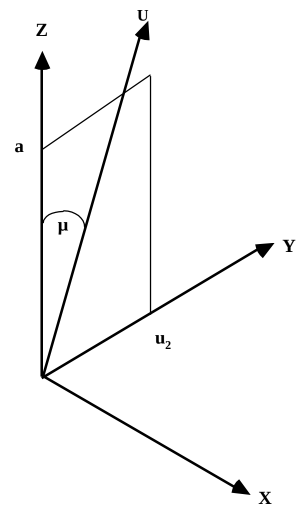




After $R_y(-ß)$, angle μ lies in the y-z plane

3. Rotate -μ degrees about the x-axis

$R_x(\mu)$



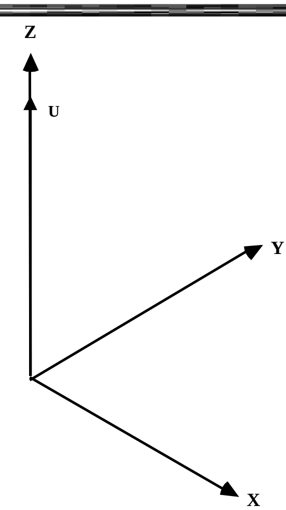
$\cos \mu = a/||u||$

$\sin \mu = u_2/||u||$

4. Rotate R degrees about the z-axis

U is aligned with the z-axis

Apply the original rotation, **R**



5. Apply the inverses of the transformations in reverse order.

Rotation About an Arbitrary Axis

- $T^{-1}\ R_y(\beta)\ R_x(-\mu)\ \ R\ \ R_x(\mu)\ R_y(-\beta)\ T$

Alternate view of the Rotation Matrix

Given P_1, P_2, P_3

P_1P_2 is direction, P_1P_3 is "up"

$$R_{cc} = \begin{pmatrix} r_{1x} & r_{2x} & r_{3x} & 0 \\ r_{1y} & r_{2y} & r_{3y} & 0 \\ r_{1z} & r_{2z} & r_{3z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Alternate view of the Rotation Matrix

Z axis rotates to be aligned with P_1P_2

$R_z = [\mathbf{r}_{3x}, \mathbf{r}_{3y}, \mathbf{r}_{3z}]$ = normalized P_1P_2

X axis rotates to be normal to P_1, P_2, P_3 plane

$R_x = [\mathbf{r}_{1x}, \mathbf{r}_{1y}, \mathbf{r}_{1z}]$ = normalized $P_1P_3 \times P_1P_2$

Y axis rotates to be normal to R_x R_z plane

$R_y = [\mathbf{r}_{2x}, \mathbf{r}_{2y}, \mathbf{r}_{2z}]$ = normalized $R_z \times R_x$